Slanted Volume of Revolution

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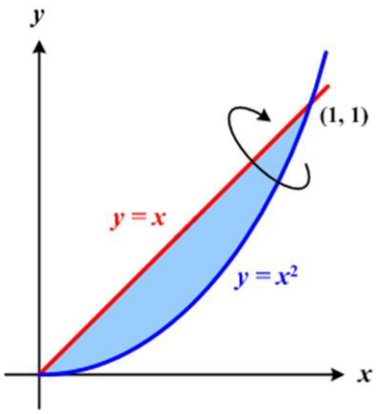

This is a fairly challenging problem[1] from BL Math Games.

Find the volume of the solid obtained by rotating the region enclosed by $y = x$ and $y = x^2$ about the line $y = x$.

My Solution

Letting r represent the perpendicular distance from the line to the parabola and s the distance along the line (Figure 1), then we are trying to evaluate the integral with respect to s as s varies from 0 to √2 given by

$$\int_0^{\sqrt{2}} \pi r^2 ds \, .$$

A standard math approach is to reduce the problem to one you know how to solve. In this case, if the planar region were rotated 45° clockwise, then it would be straight forward (Figure 1).

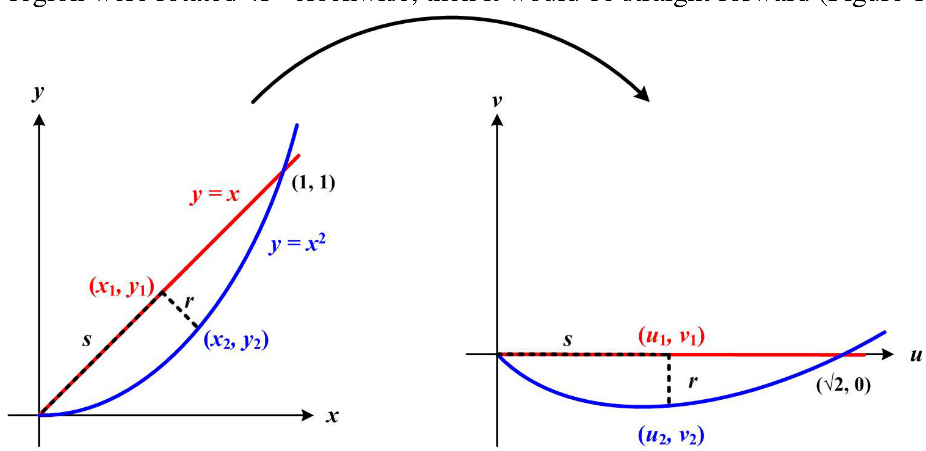

Figure 1

A rotation of planar coordinates from (x, y) to (u, v) through an angle θ measured counterclockwise from the horizontal axis is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{aligned} u &= \cos\theta x - \sin\theta y \\ v &= \sin\theta x + \cos\theta y \end{aligned}$$

So for $\theta = -\pi/4$ we have

$$u = (x + y)/\sqrt{2}$$

$$v = (-x + y)/\sqrt{2}$$

The point $(x_1, y_1) = (x_1, x_1)$ on the straight line is sent to $(u_1, v_1) = (s, 0)$, so

$$s = \sqrt{2}\, x_1$$

$$0 = (-x_1 + x_1)/\sqrt{2}$$

[1] Nov 18, 2025 (https://medium.com/math-games/can-you-do-this-slanted-revolution-calculus-puzzle-b8b5f86233b5)

And the point $(x_2, y_2) = (x_2, x_2^2)$ on the parabola is sent to $(u_2, v_2) = (s, -r)$, so

$$s = (x_2 + x_2^2)/\sqrt{2}$$

$$-r = (-x_2 + x_2^2)/\sqrt{2}$$

These last equations provide us with a parameterization of our desired variables s and r by the single variable x_2. We will drop the subscript from now on. Therefore

$$ds = \frac{ds}{dx}dx = \frac{1}{\sqrt{2}}(2x+1)dx$$

$$r^2 = \frac{1}{2}(x^2 - x)^2 = \frac{1}{2}(x^4 - 2x^3 + x^2)$$

So integrating with respect to s from 0 to $\sqrt{2}$ (and with respect to x from 0 to 1), we have the volume of rotation is given by

$$\pi\int_0^{\sqrt{2}} r^2 ds = \frac{\pi}{2\sqrt{2}}\int_0^1 (2x^5 - 3x^4 + x^2)dx = \frac{\pi}{2\sqrt{2}}\left(\frac{2}{6}x^6 - \frac{3}{5}x^5 + \frac{1}{3}x^3\right)\Bigg|_0^1$$

and so

$$\boxed{\pi\int_0^{\sqrt{2}} r^2 ds = \frac{\pi}{30\sqrt{2}}}$$

(Given my propensity of arithmetic errors, I have little confidence in this answer.)

BL's Approach and Comments

This was BL's approach:

Our approach would be to write our final volume through an approximation as a Riemann sum and take a limit of that approximation to arrive at our answer.

Wow. That seems very hard. The calculations were behind the subscription wall, so I couldn't tell if his approach ended with an integral to evaluate to yield a closed form answer or if it led to only a numerical answer.

Actually, one of the commenters, Venkat Krishnan, followed my approach and got the same answer as I did. Yeah!

The exact answer is pi * sqrt(2)/60 = 0.07405.... One way to do it is a change in coordinate using:

u = (x + y)/sqrt(2) and v = (x - y)/sqrt(2)

This rotates everything 45 degrees so the y = x line becomes v = 0. Then integrate using the disk method. Not easy. The method from the article gets you very close.

—Venkat Krishnan[2]

[2] https://medium.com/@pk.venkataraghavan