A Simple Ratio

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$$\frac{1+3+5+...+2013+2015}{2+4+6+...+2014+2016}$$

This is a straight-forward problem[1] by Ritvik Nayak from the Puzzle Sphere.

Evaluate the ratio. It's actually simpler than you might think.

Apparently it is a sample problem from the Southeast Asian Math Olympiad (SEAMO).

My Solution

$$R = \frac{1+3+5+...+2013+2015}{2+4+6+...+2014+2016} = \frac{\sum_{k=1}^{1008}(2k-1)}{\sum_{k=1}^{1008}2k} = \frac{\sum_{k=1}^{1008}2k - 1008}{\sum_{k=1}^{1008}2k}$$

Let $$S = \sum_{k=1}^{1008}2k = 2\sum_{k=1}^{1008}k = 2\frac{1008\cdot 1009}{2} = 1008\cdot 1009$$

Then $$R = \frac{S-1008}{S} = \frac{1008\cdot 1009 - 1008}{1008\cdot 1009} = \frac{1008}{1009}$$

Ritvik Nayak Solution

This is essentially the same solution but from a different angle. I feel it relies on more memorization than mine.

To solve this problem, first we need to break it down into 2 equations :

(1 + 3 + 5…+ 2013 + 2015) and (2 + 4 + 6… + 2014 + 2016)

To solve the problem, we need to first find the total amount of numbers in each equation. Instead of individually counting the numbers (which would take a *very very* long time), we can do something much smarter. First, let's find the total number of terms in the sequence — (1 + 3 + 5…+ 2013 + 2015). This is an arithmetic series where the first term ***a*** = 1 and the common difference ***d* = 2**, to find the total number of terms (n) in this series, we can use the formula for the **(*n*)**-th term of an arithmetic sequence:

$$a_n = a + (n - 1)d$$

If we set (a_n) ***as* 2015*, then we can concur that :***

$$2015 = 1 + (n - 1)\cdot 2$$

$$\therefore 2015 = 1 + 2n - 2,$$

so* 2015 = 2*n* − 1*, this means that* 2016 = 2*n

$$\therefore n = 1008.$$

[1] 14 July 2024 (https://medium.com/puzzle-sphere/can-you-solve-this-southeast-asian-math-olympiad-seamo-sample-problem-898e92c3de82)

1008 is thus the total number of terms in the series (1 + 3 + 5…+ 2013 + 2015). Now to actually calculate the sum of the series, we can apply Gauss Summation:

$$1008 \cdot (2015 + 1) = 1008 \cdot 2016 = 1016064 \; [= 1008 \cdot 1008]$$

1016064 is the total sum of the series.

From Wikipedia,

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) = n\,(2a + (n - 1)d)/2 = n\,(a + a_n)/2.$$

So the following result should be

$$(1008/2) \cdot (2015 + 1) = 504 \cdot 2016 = 1016064$$

Since the final result 1016064 is the same, the missing 2 and the 1008 must be a dreaded transcription error we all experience from time to time. And he gets the formula correct in the next computation.

My first quibble with this is that it requires the memorization of a more general formula for the "Gaussian Summation" than the simple $1 + 2 + \ldots + n = n(n + 1)/2$. Yes, it saves time manipulating more complex expressions involving an arithmetic progression, but as my solution shows, they are trivial manipulations which any student should be able to handle if they understand summations.

We can apply this to the next series — (2 + 4 + 6… + 2014 + 2016).

$$a_n = a + (n - 1)d$$

$$a = 2 \;\&\; d = 2$$

$$a_n = 2016$$

$$2016 = 2 + (n - 1) \cdot 2,$$

therefore*, 2016 = 2 + 2*n* − 2, *so* 2016 = 2*n

$$\therefore n = 1008$$

$$(1008/2) \cdot 2018 = 504 \cdot 2018 = 1018032 \; [= 1008 \cdot 1009]$$

Now that we have the two sums of the series, we can solve the equation.

1016064 / 1018032 is the answer, however, it can be simplified further into 1008 / 1009, the final answer.

My second quibble is that multiplying out of numbers leads to larger results that just have to be reduced later. Also, it is easier to keep numbers factored, since, as in this case, the factors may cancel in later computations. And furthermore, fewer arithmetic mistakes are made when handling smaller numbers.