Interest Calculations

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A recent video[1] by Angela Collier about compound interest reminded me of an effort I did years ago to derive the formulas for interest calculations, especially for installment payments on a loan like a mortgage. In the process I showed where Euler's constant e can show up. I thought I would resurrect the material, even though I imagine modern texts now provide the information (I couldn't easily locate it back then).

Interest. Let P be the amount borrowed from the lender, called the principal, and let T be the duration of the loan. Then interest I is the charge the loaner imposes on the borrower for the use of the money. Typically if the amount of money loaned doubles, then the interest should double, and if the length of time doubles, then again the interest should double, or if the time halves, then the interest should halve. So we are assuming the interest varies linearly in both principal and time. We saw in the post "Fibonacci, Chickens, and Proportions"[2] that this implies interest is modeled by

$$I = rPT$$

for some constant r called the rate of interest.

Typically interest rates are given as annual rates, that is, the number of dollars of interest charged per dollars of principal per year. In fact, they are given as percentages—dollars of interest per 100 dollars of principal per year. Therefore an annual rate R of 6% yields 6 dollars of interest for every 100 dollars of principal per year.

Compound Interest. What we have been discussing up to now is simple interest where the borrower is only paying interest on the principal. The loaner is essentially deferring receiving the return on the money until the end of the loan when the principal and interest are all paid back. So in effect the loaner is losing the use of more money over time (the interest) than just the principal. In order to compensate for this loss, interest is added to the loan at discrete time intervals and subsequent interest is charged on this increasing amount. This is called compound interest.

For example, suppose the principal P_0 is \$1000 and the annual interest rate R is 6%. Suppose the interest is compounded semi-annually, that is, after 6 months the interest is added to the principal, and then after 12 months ($T = 1$ year) the interest on the principal and the 6-month interest is added to the principal and 6-month interest. This is shown algebraically as follows.

First, compute the interest for 6 months or half a year on the principal: $I = RPT/2 = 0.06\cdot 1000\cdot ½$. Then the amount of the loan P_1 after 6 months is (principal + interest):

$$P_1 = P_0 + (R/2)P_0 = (1 + R/2)P_0 = (1.03)1000 = \$1030.00$$

Repeating this computation after another 6 months, namely a year, yields a final value P_2 of the loan of

$$P_2 = P_1 + (R/2)P_1 = (1 + R/2)P_1 = (1 + R/2)^2P_0 = (1.03)^2\ 1000 = \$1060.90,$$

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[1] "Brought to you by the number e", 15 January 2026 (https://www.youtube.com/watch?v=JJz5D9txeGA&pp=0gcJCYcKAYcqIYzv)

[2] https://josmfs.net/wordpress/2019/09/06/fibonacci-chickens-and-proportions/

which only nets \$0.90 more than the simple interest one-year loan. If the one-year loan were compounded monthly, then the final value P_{12} at the end of a year would be

$$P_{12} = (1 + R/12)^{12} P_0 = (1.005)^{12}\ 1000 = \$1061.68$$

Not much of a difference, netting \$1.68.

Exponential Limit. What's going on? We are interested in the term $(1 + R/k)^k$ as k gets arbitrarily large, that is, as $k \to \infty$. Now for $u = R/k$

$$(1 + R/k)^k = [(1 + u)^{1/u}]^R$$

So what is the limit of $(1 + u)^{1/u}$ as $u \to 0$?

Let $y = (1 + u)^{1/u}$. Then $\ln y = \ln(1 + u) / u$, which goes to 0/0 as $u \to 0$, an indeterminate form. Therefore we apply L'Hospital's Rule for differentiable functions f and g, where $f(0) = g(0) = 0$:

$$\lim_{u\to 0} \frac{f(u)}{g(u)} = \lim_{u\to 0} \frac{f'(u)}{g'(u)}$$

So as $u \to 0$

$$\ln y = \ln(1 + u) / u \ \to\ (1 + u)^{-1} / 1 \ \to\ 1$$

Therefore as $u \to 0$

$$y = (1 + u)^{1/u} \ \to\ e \qquad \text{(Euler's constant)}$$

Or as $n \to \infty$

$$(1 + 1/n)^n \ \to\ e$$

which is the more standard version. (I should note this is circular reasoning, since $(1 + u)^{1/u} \to e$ is used in one of the derivations of the derivative of $\ln x$. This is not surprising, since $\ln x = \log_e x$.)

Now $e = 2.71828\ldots$, so as $k \to \infty$,

$$(1 + R/k)^k \to\ e^R \approx (2.71828)^{.06} = 1.06184$$

Therefore if interest is compounded instantaneously, the final amount of the one-year loan doesn't grow without bound. Instead it won't exceed $e^R P_0$, netting a max of \$1.84 over the simple interest loan. And thus e enters into interest calculations.

Example. Notice that if the time period for the loan is more than one year, say $T = 10$ years, with the same monthly compounding, then

$$P_{120} = (1 + R/12)^{12} P_{108} = \ldots = [(1 + R/12)^{12}]^{10} P_0 \approx (1.06168)^{10}\ P_0 = (1.81940)\ P_0$$

or for instantaneous compounding

$$P_{10\text{ years}} = (e^R)^{10}\ P_0 \approx (1.82212)\ P_0.$$

So for a 30-year mortgage,

$$P_{30\text{ years}} = [(1 + R/12)^{12}]^{30} P_0 = (1.06168)^{30}\ P_0 = (6.02258)\ P_0.$$

The interest alone would be five times the original principal. This is the effect of exponential growth. The borrower can't wait 30 years to pay back the loan in a lump sum. They need to start paying it back sooner.

Installment Payments

What if we take out a loan of P_0 dollars at annual interest rate R for a period of $T = n$ years, compounded monthly, and we must make equal monthly payments x so that the loan is fully paid off in T years. What would these monthly installments x be? Examples would be mortgages and car loans.

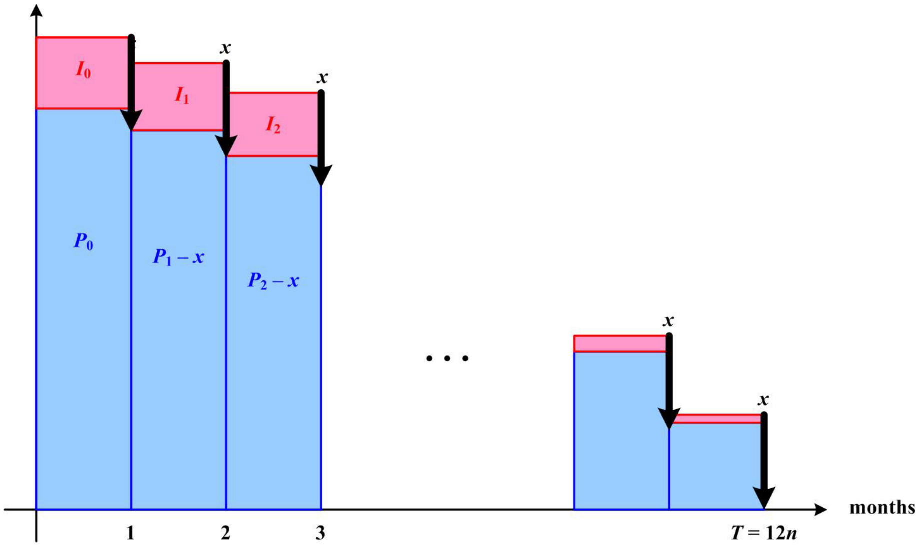

Figure 1

Figure 1 shows the situation. Initially we owe P_0 dollars. After the first month we still owe P_0 dollars, but now we also owe interest I_0 for one month on P_0 dollars, namely $I_0 = RP_0(1/12) = (R/12)P_0$ dollars. Therefore, after one month we owe

$$P_1 = P_0 + I_0 = P_0 + \tfrac{R}{12} P_0 = \left(1 + \tfrac{R}{12}\right) P_0 .$$

So we pay our first installment x. Then we owe $P_1 - x$ dollars.

After the second month we still owe $P_1 - x$ dollars, but now we owe interest on it for one month, namely $I_1 = (R/12)(P_1 - x)$ dollars. Therefore, after two months we owe

$$\begin{aligned} P_2 &= (P_1 - x) + \tfrac{R}{12}(P_1 - x) = (P_1 - x)\left(1 + \tfrac{R}{12}\right) \\ &= P_1\left(1 + \tfrac{R}{12}\right) - x\left(1 + \tfrac{R}{12}\right) \\ &= P_0\left(1 + \tfrac{R}{12}\right)^2 - x\left(1 + \tfrac{R}{12}\right) \end{aligned}$$

So we pay our next installment x and now owe $P_2 - x$ dollars.

After the third month we owe

$$\begin{aligned} P_3 &= (P_2 - x) + \tfrac{R}{12}(P_2 - x) = (P_2 - x)\left(1 + \tfrac{R}{12}\right) \\ &= \left(P_0\left(1 + \tfrac{R}{12}\right)^2 - x\left(1 + \tfrac{R}{12}\right) - x\right)\left(1 + \tfrac{R}{12}\right) \\ &= P_0\left(1 + \tfrac{R}{12}\right)^3 - x\left(\left(1 + \tfrac{R}{12}\right)^2 - \left(1 + \tfrac{R}{12}\right)\right) \end{aligned}$$

So we pay x dollars and now owe $P_3 - x$ dollars. And so on.

So after m months we owe

$$P_m = P_0\left(1 + \tfrac{R}{12}\right)^m - x\left(\left(1 + \tfrac{R}{12}\right)^{m-1} + \left(1 + \tfrac{R}{12}\right)^{m-2} + \ldots + \left(1 + \tfrac{R}{12}\right)^2 + \left(1 + \tfrac{R}{12}\right)\right)$$

Evaluating the partial sum of the geometric series yields

$$P_m = P_0\left(1+\tfrac{R}{12}\right)^m - x\left[\frac{1-\left(1+\tfrac{R}{12}\right)^m}{1-\left(1+\tfrac{R}{12}\right)} - 1\right] \tag{1}$$

or

$$P_m = P_0\left(1+\tfrac{R}{12}\right)^m - \frac{x}{\frac{R}{12}}\left[\left(1+\tfrac{R}{12}\right)^m - 1\right] + x \tag{2}$$

Now we want to choose the installment x so that after $T = n$ years $= 12n$ months we will owe x dollars. That is, our last installment x will pay off the loan and we will owe 0 dollars. Thus $P_{12n} = x$. Substituting this into equation (2), we get

$$x = P_{12n} = P_0\left(1+\tfrac{R}{12}\right)^{12n} - \frac{x}{\frac{R}{12}}\left[\left(1+\tfrac{R}{12}\right)^{12n} - 1\right] + x$$

Canceling the x's and collecting terms, we get

$$\frac{x}{\frac{R}{12}}\left[\left(1+\tfrac{R}{12}\right)^{12n} - 1\right] = P_0\left(1+\tfrac{R}{12}\right)^{12n}$$

or finally

$$\boxed{x = \frac{\frac{R}{12}}{1-\left(1+\tfrac{R}{12}\right)^{-12n}}P_0} \tag{3}$$

Example. So if we have a mortgage for P_0 = \$400,000 at a 6% annual interest rate R for n = 30 years, then our monthly mortgage payments x would be

$$x = (0.0059955)P_0 = (0.0059955)(400{,}000) = \$2398.20.$$

Comment 1. If we think of r (= $R/12$) as the interest rate for an installment interval, then equation (3) can be simplified to

$$\boxed{x = \frac{r}{1-(1+r)^{-m}}P_0} \tag{4}$$

so that x is the installment payment needed to pay off the purchase P_0 in m installments at an installment interval interest rate of r.

Comment 2. Now instead of using simple interest for the installment interval, $(1 + r)$, we could compound the interest in that interval over k subintervals: $(1 + r/k)^k$. So as we argued above (Exponential Limit), if we compound instantaneously, the principal plus interest is e^rP instead of $(1 + r)P$. Then substituting e^r for $(1 + r)$ in equation (1), equation (4) becomes for instantaneous compound interest

$$x = \frac{e^r - 1}{1-e^{-rm}}P_0 \tag{5}$$

Then in the case of equation (3), where $r = R/12$ and $m = 12n$, we get

$$x = \frac{e^{\frac{R}{12}} - 1}{1-e^{-Rn}}P_0 \tag{6}$$

So for the mortgage of P_0 = \$400,000 at a 6% annual interest rate R for $n = 30$ years, our monthly mortgage payments x would be

$$x = (0.0060052)P_0 = (0.0060052)(400{,}000) = \$2402.07,$$

which is naturally a little more (\$3.87) than the installment we got before, since instantaneous compounding means slightly more interest is charged each month.

Now $e^r - 1 = r + r^2/2! + r^3/3! + \ldots$. So for small r, $e^r - 1 \approx r$ and equation (5) becomes

$$x = \frac{r}{1 - e^{-rm}} P_0$$

And for $r = R/12$ (and $m = 12n$), equation (6) becomes

$$x = \frac{\frac{R}{12}}{1 - e^{-Rn}} P_0$$

Using this formula for the mortgage of P_0 = \$400,000 at a 6% annual interest rate R for $n = 30$ years, our monthly mortgage payments x would be

$$x = (0.0059902)P_0 = (0.0059902)(400{,}000) = \$2396.07,$$

which is a little less (\$2.13) than the original installment, since $e^r - 1$ is a little greater than the approximation r.