Three Circles Problem

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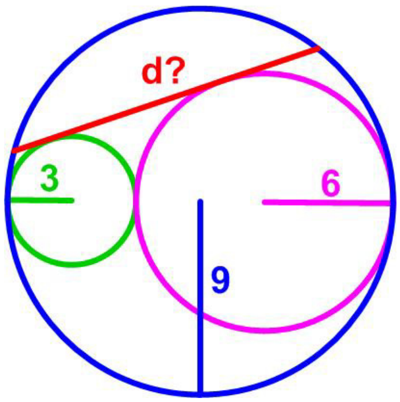
Jim Stevenson



This is a problem from the 1995 AIME problems ([1]).

Circles of radius 3 and 6 are externally tangent to each other and are internally tangent to a circle of radius 9. The circle of radius 9 has a chord that is a common external tangent of the other two circles. Find the square of the length of this chord.

My Solution

Parameterize the problem as shown in Figure 1. (Use the facts that radii of circles are perpendicular to tangents to circles at the point of tangency, that the perpendicular bisector of a chord of a circle passes through its center, and that the line joining the centers of two tangent circles pass through their point of tangency.) Then by similar triangles

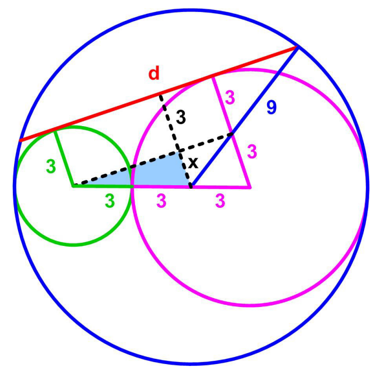


Figure 1

$$x/6 = 3/9 = 1/3$$

so that $x = 2$. Then by the Pythagorean Theorem.

$$9^2 = (d/2)^2 + (3 + x)^2 = (d/2)^2 + 5^2$$

so $$d^2 = 4(9^2 - 5^2) = 4(9 - 5)(9 + 5) = 4\cdot4\cdot14$$

$$= 16\cdot14 = (15 + 1)(15 - 1) = 225 - 1 = 224$$

(I really like to avoid arithmetic with large numbers.)

AIME Solution (Analytic Geometry)

AIME had two solutions. The first was essentially the same as mine. The second used analytic geometry.

Let A be defined as the origin of a coordinate plane with the x-axis running across the chord [Figure 2] and $C(6\sqrt{2}, 0)$ by the Pythagorean Theorem.[1] Then we have $D(0, -6)$ and $F(6\sqrt{2}, -3)$, and since DE/DF = 1/3, the point E is one-third of the way from D to F, so point E has coordinates $(2\sqrt{2}, -5)$. E is the center of the circle with radius 9, so the equation of this circle is

$$(x - 2\sqrt{2})^2 + (y + 5)^2 = 81.$$

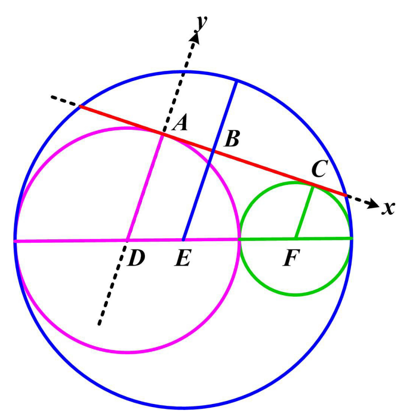


Figure 2

[1] JOS: I guess one way to do this is similar to what I did. Draw a line parallel to the x-axis through F and intersecting the y-axis. Since CF = 3, then this line bisects AD = 6 into segments of length 3. Therefore by the Pythagorean Theorem

$$(AD/2)^2 + AC^2 = DF^2 \text{ or } 3^2 + y^2 = (6 + 3)^2 = 81$$

So $$y^2 = 72 = 36\cdot2 \text{ or } y = 6\sqrt{2}.$$

Since the chord's equation is $y = 0$, we must find all values of x satisfying the equation of the circle such that $y = 0$. We find that

$$x - 2\sqrt{2} = \pm\sqrt{56},$$

so the chord has length

$$|(\sqrt{56} + 2\sqrt{2}) - (-\sqrt{56} + 2\sqrt{2})| = 2\sqrt{56}$$

and the answer is $(2\sqrt{56})^2 = \boxed{224}$.

References

[1] "Problem 4" 1995 AIME Problems (https://artofproblemsolving.com/wiki/index.php/1995_AIME_Problems)