Functional Equation Puzzle

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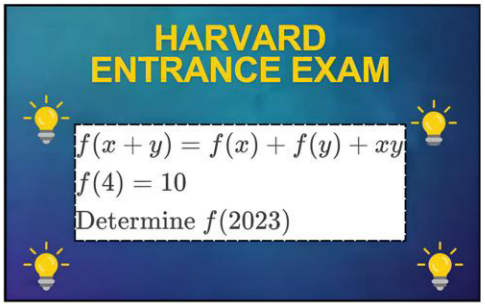

This is a math Olympiad problem[1] from Puzzle Sphere where Muhammad Zain Sarwar claims it is at Harvard entrance exam level.

Given the functional relationship $f(x + y) = f(x) + f(y) + xy$ with the known value $f(4) = 10$, determine the value of $f(2023)$.

Just try some examples and detect the pattern that defines the function.

Solution

Consider some initial cases.

$$f(0) = f(0 + 0) = f(0) + f(0) + 0 \quad \Rightarrow f(0) = 0$$

$$f(4) = f(2 + 2) = f(2) + f(2) + 4 = 10 \quad \Rightarrow f(2) = 3$$

$$f(2) = f(1 + 1) = f(1) + f(1) + 1 = 3 \quad \Rightarrow f(1) = 1$$

$$f(3) = f(2 + 1) = f(2) + f(1) + 2 = 3 + 1 + 2 \quad \Rightarrow f(3) = 6$$

$$f(5) = f(4 + 1) = f(4) + f(1) + 4 = 10 + 1 + 4 \quad \Rightarrow f(5) = 15$$

So we get the following table

n	$f(n)$	$f(n) - f(n-1)$
0	0	
1	1	1
2	3	2
3	6	3
4	10	4
5	15	5
…		

Claim: $\boldsymbol{f(n) - f(n - 1) = n}$.

Proof. Since $f(x + y) = f(x) + f(y) + xy$ and $f(1) = 1$, we have

$$\boldsymbol{f(n) - f(n-1)} = f(n - 1 + 1) - f(n - 1) = f(n - 1) + f(1) + 1\cdot(n - 1) - f(n - 1) = 1 + (n - 1) = \boldsymbol{n}$$

Therefore, since $f(0) = 0$,

$$\boldsymbol{f(n)} = [f(n) - f(n - 1)] + [f(n - 1) - f(n - 2)] + [f(n - 2) - f(n - 3)] + \ldots + [f(2) - f(1)] + [f(1) - f(0)]$$

$$= n + (n - 1) + (n - 2) + (n - 3) + \ldots + 2 + 1$$

$$= \boldsymbol{n(n + 1)/2}$$

[1] 10 October 2025 (https://medium.com/puzzle-sphere/only-geniuses-can-solve-this-harvard-entrance-exam-level-functional-equation-054fe8476a41)

And so,

$$f(2023) = 2023 \cdot 2024/2 = 2{,}047{,}276$$

(I don't know Puzzle Sphere's solution since it is behind a subscription wall.)