Triangle Bow-tie Problem

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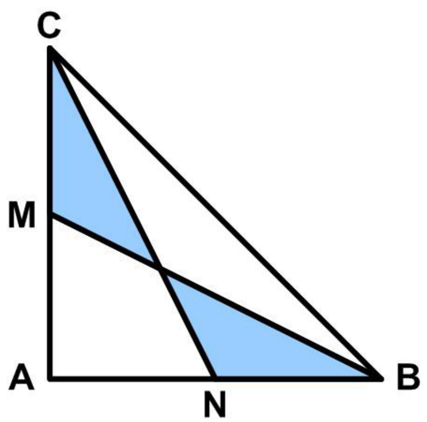

This is another problem from Dan Griller ([1]).

In the triangle ABC, CN and MB are straight lines, ∠CAB = 90° and CM = MA = AN = NB = 5. Find the exact area of the shaded region.

My Solution

I resorted to an analytic geometry solution rather than a plane geometry one. As shown in Figure 1, the areas labeled E, F, G, and H satisfy

$$50 = E + F = G + H \quad \text{and}$$

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Therefore, subtracting the two equations we get F = G, and so E = H.

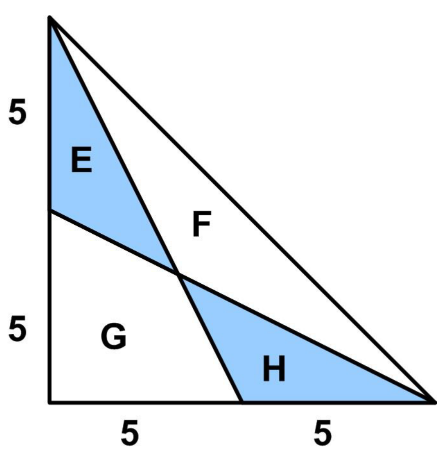

Figure 1

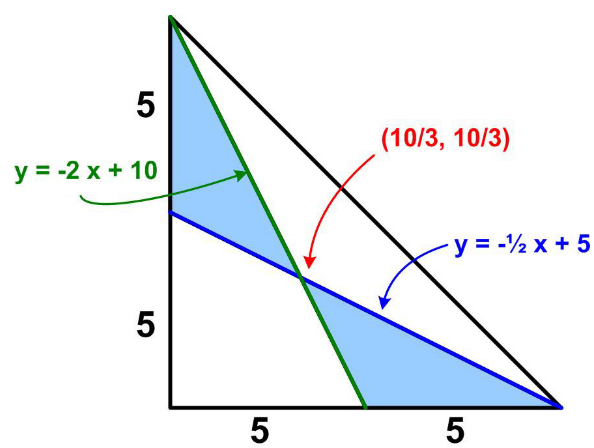

Figure 2

Figure 2 shows the crossing lines have linear equations

$$y = -2x + 10 \text{ and}$$

$$y = -\tfrac{1}{2}x + 5.$$

Solving them simultaneously yields the point of intersection to be $(^{10}/_3,\ ^{10}/_3)$. Therefore the altitude for the triangle with area E is $x = {}^{10}/_3$, which means the total area of the blue triangles is

$$E + H = \tfrac{1}{2}\, 5\, (^{10}/_3) + \tfrac{1}{2}\, 5\, (^{10}/_3) = {}^{50}/_3.$$

Griller Solution

Griller has a pure plane geometry solution.

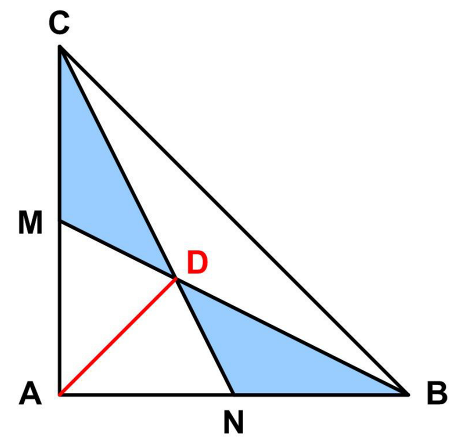

Figure 3

Let […] denote area. We have [Figure 3]

$$[ANC] = \tfrac{1}{2} \times 5 \times 10 = 25 = [ABM]$$

and so, since the region ANDM is common to ΔANC and ΔABM, it follows that the two blue triangles also have equal areas, say

$$[CDM] = [BDN] = x$$

Now, ΔAND and ΔBND have the same base and same height, so they have the same area x. So we have

$$3x = [ANC] = 25,$$

hence

$$x = {}^{25}/_{3}$$

Thus the area of the blue region is

$$2x = {}^{50}/_{3}.$$

References

[1] Griller, Daniel, *Elastic Numbers: 108 Puzzles for the Serious Problem Solver*, Rational Falcon, 2017. Diamond Problem #26. (Scale of difficulty: Bronze, Silver, Gold, Diamond.)