Yet Another Sum

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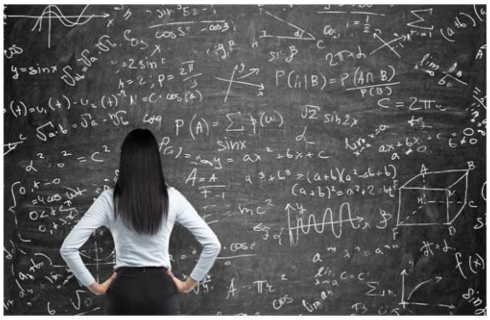

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This is another challenging sum from the 2024 Math Calendar ([1]).

Find x where $x = e^t$ and

$$t = \sum_{n=1}^{\infty} \frac{1}{n2^{n-2}}$$

As before, recall that all the answers are integer days of the month.

Solution

Again we approach the answer via power series, this time of the form

$$P(x) = 4\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Then $t = P(½)$. Again we use the geometric series

$$G(x) = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

This time instead of differentiating, we integrate term by term to get (for $0 \le x < 1$)

$$\int_0^x G(r)dr = r + \frac{r^2}{2} + \frac{r^3}{3} + \ldots \Bigg|_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{1}{4}P(x) = \int_0^x \frac{1}{1-r}dr = -\ln(1-r)\Big|_0^x = -\ln(1-x)$$

So

$$t = P(\tfrac{1}{2}) = 4\sum_{n=1}^{\infty} \frac{1}{n2^n} = -4\ln(1-\tfrac{1}{2}) = 4\ln 2$$

Therefore

$$\boxed{e^t = e^{4\ln 2} = 2^4 = 16}$$

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2024: Your Daily epsilon of Math*, American Mathematical Society, 2024. November