Circular Ant Problem

23 February 2025

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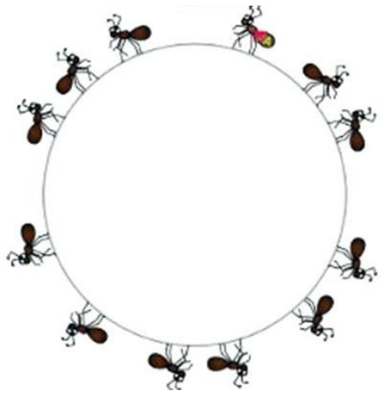

This is a wicked variation of the ant problem on a stick[1] by Peter Winkler.

Twenty-four ants are randomly placed on a circular track of length 1 meter; each ant faces randomly clockwise or counterclockwise. At a signal, the ants begin marching at 1 cm/sec; when two ants collide they both reverse directions. What is the probability that after 100 seconds, every ant finds itself exactly where it began?

My Solution

I am going to use spacetime diagrams similar to the ones in the ant problem on a stick. To capture the periodicity of the circle in a linear diagram, I will repeat one meter sections on each side of the main one meter view. Figure 1 shows what counterclockwise and clockwise paths would look like in this view. If unobstructed, each 1 cm/sec moving ant would return to its initial position after 100 seconds, after having traveled 1 meter. I would like to call this the starting ant's "base path".

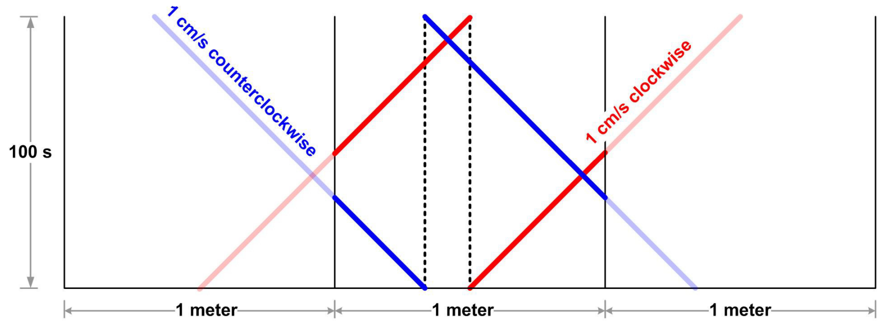

Figure 1

Two Ants. To see what is going on, let's start with a simple case of two ants. If the ants are moving in the same direction, they won't collide and after 100 seconds they will be back to their original locations (Figure 2).

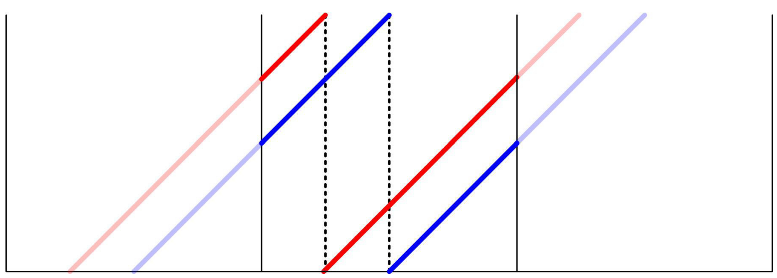

Figure 2

Now suppose the two ants move in opposite directions (Figure 3). Then their base paths will cross each other's base paths twice. That means there will be two collisions each (Figure 4).

[1] "Ant Problem" (http://josmfs.net/2018/12/28/ant-problem/)

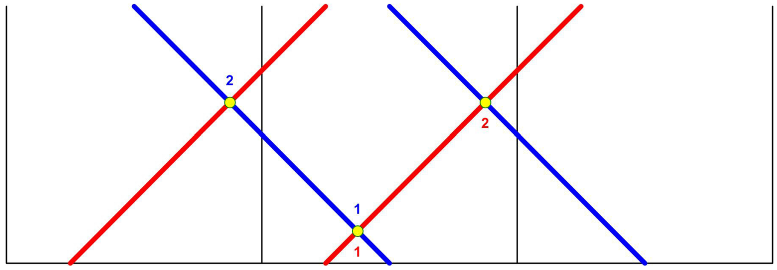

Figure 3

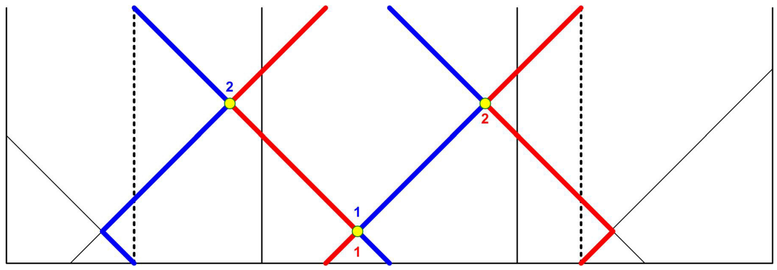

Figure 4

Notice that after the second collision the ants are back on their respective base paths and will return to their original positions.

Three Ants. Now consider three ants. Suppose they all move in the same direction. Then again they will arrive at their original positions after 100 seconds (Figure 5).

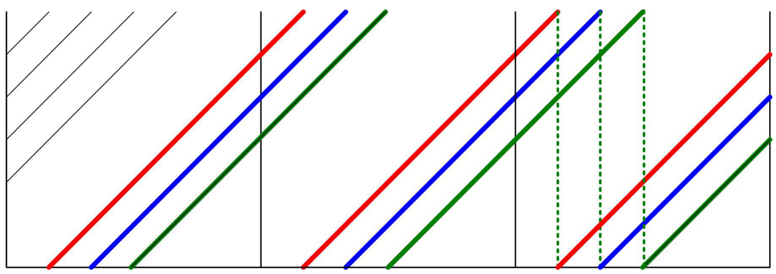

Figure 5

Suppose one ant is moving in the opposite direction from the other two (Figure 6). Then that ant's base path will intersect the other two base paths twice, or four times in all. The original right-moving (clockwise) ants' base paths will each intersect the left-moving (counterclockwise) ant's path twice.

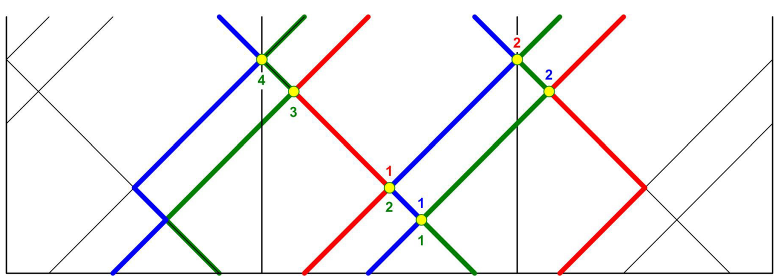

Figure 6

Notice what happens as we travel along the left-moving ant's base path. Each collision will involve the next ant to the left of this ant, in the original order of the ants on the circle. So after three collisions, the original ant will be back on its base path. But there is one more collision, and so it is diverted from its base path and does not arrive at its original position. However, it will end up on the base path of the ant it collided with. Therefore all the ants arrive at original ant positions, but not necessarily their own.

Note further that every leftward leaning base path will intersect the same number of rightward leaning base paths no matter what it's original position is. That is, sliding a leftward leaning path along the baseline will always intersect the same number of rightward leaning paths. Therefore, we get the same result regardless of which of the three ants is moving leftward.

Four Ants. Now consider four ants. Again if all four ants are moving in the same direction, they will return to their original locations after 100 seconds. Suppose one ant is moving to the left (Figure 7). Then its base path will intersect the base paths of the other three ants twice, or six times in all. But the three right-moving ants' base paths will each intersect the left-moving ant's base path twice.

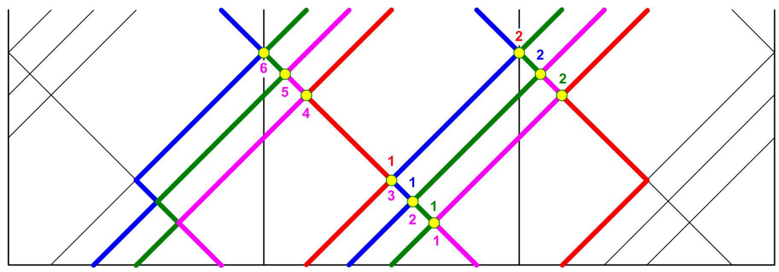

Figure 7

Again we see after four collisions the left-moving ant is back on its base path. But there are two more collisions, which again divert the left-moving ant so it does not reach its original position.

Suppose two of the four ants are moving left (Figure 8). Then something interesting happens. The base paths of these two ants each intersect the base paths of the two right-moving ants twice each or four times. That means each of the left-moving ants are back on their base paths. But the two right-moving ants' base paths are also intersecting the two left-moving base paths twice each or four times, so that they also return to their base paths. This means all four of the ants return to their original locations.

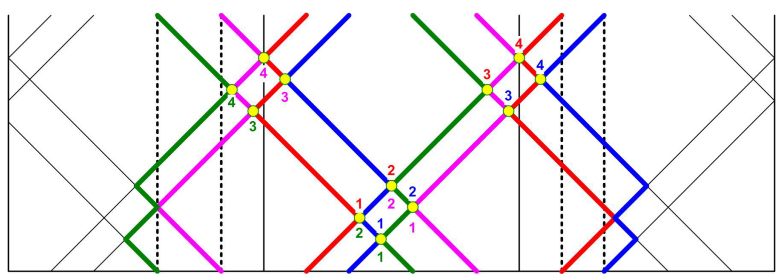

Figure 8

Five Ants. We already know if all five ants are moving in the same direction, they will return to their original locations. So consider what happens if one of the five ants is moving in the other

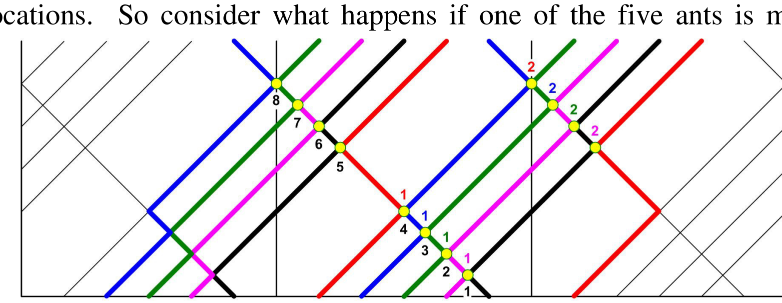

Figure 9

direction (Figure 9).

Again the left-moving ant is back on its base path after 6 collisions, but then diverted by the subsequent 2 collisions. So none of the ants arrive at their original locations.

We see that when one ant switches its original direction of motion, its base path always intersects an *even* number of opposite moving base paths, and these opposite moving paths always each intersect the original path twice. Now an initially moving ant can only return to its base path after there have been n collisions, where n equals the total number of ants. If n is odd, the ants will never ultimately return to their base paths and so arrive at their initial locations.

Six Ants. So how about an even number of ants. Will they always return to their starting positions? As a final example, consider 6 ants with two ants moving to the left and four moving to the right. The left-moving ants are back on their base paths after 6 collisions, but diverted by the subsequent two collisions. The right-moving ants' base paths only intersect the left-moving paths 4 times, and so the right-moving ants never get a chance to return their base paths. So even an even-number of ants case fails.

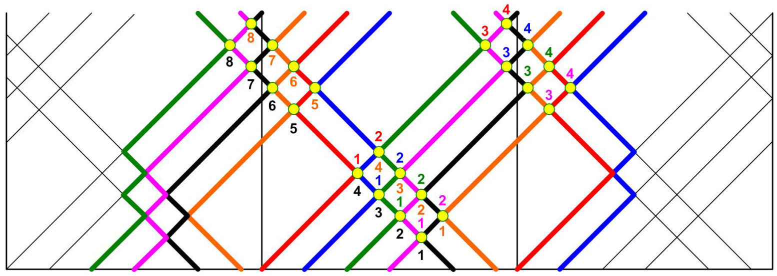

Figure 10

However, it is easy to see, as in the case of four ants, that if three ants move to the left (and so three move to the right), there would be 6 collisions each, and so they would all return to their base paths and arrive at their initial locations.

Conclusion. We see there are three and only three cases when ants return to their initial locations:

1. when they all move to the right (clockwise);
2. when they all move to the left (counterclockwise); and
3. when there is the same number of ants moving to the right as to the left (therefore an even number of total ants).

Combinatorial Calculations. So now we need to calculate for an even number of ants n how many cases are there where half the ants are moving to the right (and so half are moving to the left). Combinatorics is my least favorite exercise.

The total possibilities for initial movements of all n ants, given that each ant has two choices (left or right), is 2^n. Suppose all the ants are moving to the right. Suppose we choose 3 ants to move to the left. Initially, there are n choices for an ant to be moving to the left. After that there are $(n-1)$ choices for another ant to move to the left, and then $(n-2)$ for a third ant to move to the left. Say ants numbered 10, 13, and 7 were chosen in that order. Suppose we start over and want to choose 3 ants moving to the left. Now suppose the choices happen to be the same ants, but in a different order: 13, 7, and 10. These are the same 3 ants moving to the left, so we don't have a distinct ultimate choice. Therefore, if choices are a permutation of other choices, we are ultimately counting the same result multiple times. So we want to divide the 3 choice possibilities by the number of permutations of 3 things, namely, 3!, that is, $3{\cdot}2{\cdot}1$. Now

$$\frac{n(n-1)(n-2)}{3!} = \frac{n!}{(n-3)!3!} = \binom{n}{3},$$

which is a coefficient of Pascal's Triangle or the number of ways n things can be chosen 3 at a time. So we want to compute $\binom{24}{12}$ for the current problem of 24 ants. Note that $\binom{24}{0}$ is the number of ways no ants will be moving left (they are all moving right), which is 1. And similarly $\binom{24}{24}$ is the number of ways all the ants will be moving left, namely, also 1. (Recall we define 0! = 1.)

This means the probability of 24 ants returning to their original positions is

$$P = \left[\binom{24}{0} + \binom{24}{12} + \binom{24}{24}\right] / 2^{24} = \left[1 + \binom{24}{12} + 1\right] / 2^{24}$$

I used Igor to calculate these values using an iterative procedure from Wikipedia and arrived at

$$P = .161180377$$

Winkler Solution

This time we have to be a bit more careful about the ants' anonymity; the argument that we can replace bouncing by passing tells us only that the ants' *set of locations* will be exactly the same after 100 seconds, but any particular ant might end up in some other ant's starting spot.

In fact, since the ants cannot pass one another, their final locations will be some rotation of their initial locations. Putting it another way, the whole collection will rotate by some number of ants, and we are in effect being asked to determine the probability that that number will be a multiple of 24.

In fact it could be 24 (clockwise or counterclockwise) only if the ants are all facing the same way, thus each walks once around the track without any collisions. There are 2^{24} ways to choose how the ants face of which only two have this property, so the probability of one of these out comes is a minuscule $1/2^{23}$.

Much more likely is that the net rotation will be zero. When does that happen? Well, *conservation of angular momentum* tells us that the rate of rotation of the ant collection as a whole is constant. Thus the net rotation will be zero if and only if the initial rate of rotation is zero, meaning that exactly the same number ants start off facing counterclockwise as clockwise.[2]

The probability of *that* happening is $\binom{24}{12}/2^{24}$ which is about 16.1180258%. Adding $1/2^{23}$ to that boosts the final answer to about 16.1180377%.

Carnegie Mellon University (CMU) Solution

The website The Puzzle TOAD[3] at Carnegie Mellon had a version of this puzzle, along with the ants on a stick version ([2]).

2 JOS: I have to admit I don't have a clear understanding of what Winkler is getting at here with his application of the conservation of angular momentum. A few more details, such as equations, would help. I am not sure how to relate the individual positions of the ants before and after to some type of conservation of their velocities before and after. Also the distances between the ants are varying over time, so I don't quite know what is rotating.

3 http://www.cs.cmu.edu/puzzle/index.html

Problem

Suppose now that *n* ants [traveling at a speed of one inch per second] are placed on a circle of five foot circumference and randomly choose their direction of travel and again reverse direction when they bump into each other. One of the ants is named Alice. What is the probability that Alice is back where she started, one minute after the ants start their scampering.

Solution

Imagine that each ant carries a distinct flag as it scurries along. When two ants meet they exchange flags. … Consider what happens from the point of view of the flags. After 1 minute, the flags must all end up precisely where they started. This means that the set of points on the circle where ants reside is the same after 1 minute as it was at the beginning. However the location of an individual ant is not necessarily the same. We need to figure out what happens to the ants.

Let's give all the ants numbers. Alice is 0. Clockwise from Alice is 1, etc. This ordering does not change over time. Say that initially there are R ants going clockwise and L ants going counterclockwise, so R flags move clockwise and L flags move counterclockwise. Let's consider the aggregate motion of all the ants [flags?]. In 1 minute, they'll move $5(R - L)$ feet in a clockwise direction.[4] Remember, the set of points containing ants is the same after 1 minute. This proves that after 1 minute, ant i ends up where ant $(i + R - L) \bmod n$ started. It's the only arrangement preserving the order, the placement, and the necessary distance moved.

As my diagrams show (in particular, Figure 11), the $(i + R - L) \bmod n$ shift in "positions" is correct, but not due to a rotation in which each ant ends up having been displaced the same actual *distance*. The CMU characterization is another way of putting my description.

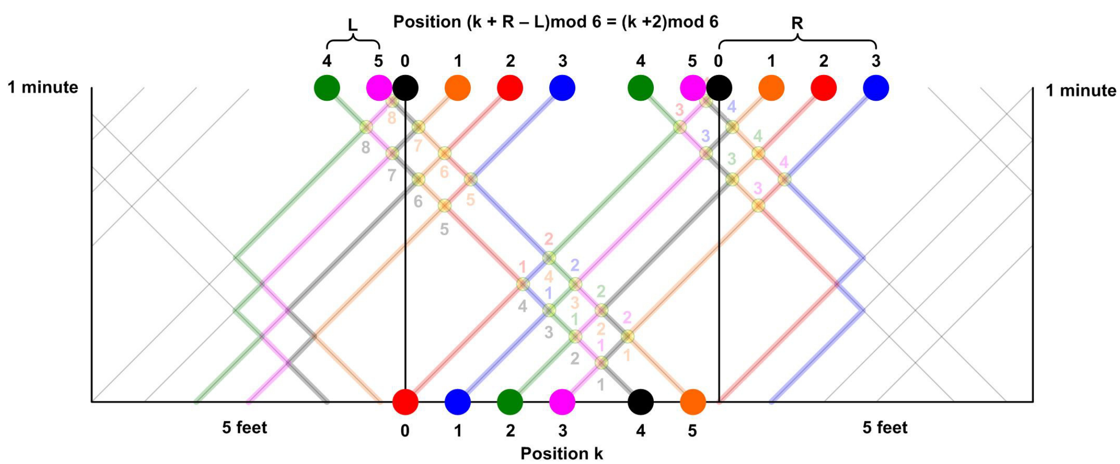

Figure 11

In fact the mapping $k \to (k + R - L) \bmod n$ can be derived from the spacetime diagrams as follows. Let k_R be the initial positions of the right-moving ants and k_L the initial positions of the left-moving ants. From Figure 11 we see that the right moving ant base path will have $2L$ collisions, at each of which the next ant to the right will take over the initial ant's base path. When the minute is up, the $2L$ ant to the right, counting cyclically, will be in the final position. Since the ants maintain their order, this means the original right moving ant will be $-2L$ positions to the left of its original starting position now occupied by the $2L$ ant to the right. So the mapping is $k_R \to (k_R - 2L) \bmod n$. In our case of $n = 6$ and $L = 2$, we have $k_R \to (k_R - 4) \bmod 6 = (k_R + 2) \bmod 6$.

[4] JOS: I have suggested that I think maybe CMU is talking about the flags here and not the ants. But I am not sure that makes any better sense. Otherwise, there is the same ambiguity about a rigid rotation as with Winkler's solution.

Similarly, the left-moving ant base path will have $2R$ collisions, at each of which the next ant to the left will take over the initial ant's base path. When the minute is up, the $-2R$ ant to the left will be in the final position. This means the original left-moving ant will end up $2R$ positions to the right of its original position. So the mapping is $k_L \to (k_L + 2R) \bmod n$. In the case of $n = 6$ and $R = 4$, we have $k_L \to (k_L + 8) \bmod 6 = (k_L + 2) \bmod 6$, the same as for the left-moving ants.

Actually, $L = n - R$ means

$$(k - 2L) \bmod n = (k - 2(n - R)) \bmod n = (k + 2R) \bmod n.$$

So we have the mapping $k_R \to (k_R + 2R) \bmod n$. Thus in general we have

$$k \to (k + 2R) \bmod n.$$

But $R = n - L$, so

$$(k + 2R) \bmod n = (k + R + R) \bmod n = (k + R + n - L) \bmod n = (k + R - L) \bmod n.$$

Therefore, we also have in general

$$k \to (k + R - L) \bmod n$$

as shown in the CMU solution.

So Alice will end up where she started after 1 minute if $R = n$ and $L = 0$ or if $R = 0$ and $L = n$ or if $R = L$ [and not otherwise]. From this we can compute our answer. Let P be the Probability that Alice ends up where she started.

$$P = \begin{cases} 2^{-n} + 2^{-n} + 2^{-n}\binom{n}{n/2} & \text{if } n \text{ is even} \\ 2^{-n} + 2^{-n} & \text{if } n \text{ is odd} \end{cases}$$

There is another way of looking at the problem: Assume w.l.o.g. that Alice moves clockwise initially. Now assume that someone rotates the circle in an anti-clockwise direction at a rate of one revolution per minute. Thus ants moving clockwise now become stationary and the others move at double speed. After one minute Alice's original position is where it was, but where is Alice? Suppose that initially there were [$R =$] k ants moving clockwise and [$L =$] $n - k$ ants moving anti-clockwise. After a collision, the ant that was moving anti-clockwise becomes stationary and the other ant moves off. Furthermore, there will be $2k(n - k)$ ant collisions altogether and so Alice will be involved in $2(n - k)$ collisions with other ants during the process. To the observer, there are always k stationary ants at positions $X_1, X_2, \ldots, X_k$ on the circle. Alice starts at X_1 and will end up at $X_{[1+2(n-k)]}$ mod n. So she will end up where she started if either $k = 0$ or $k = n$, giving the same probabilities as before.

References

[1] Winkler, Peter, "Ants on the Circle", *Mathematical Puzzles, Revised Edition*, CRC Press, 2024. p.30.

[2] "Problems with Ants", *The Puzzle TOAD*, Carnegie Mellon School of Computer Science, (http://www.cs.cmu.edu/puzzle/puzzle9.html)