Pythagorean Theorem Converse

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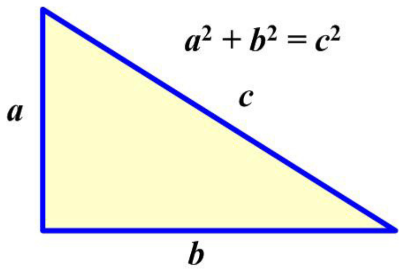
Jim Stevenson



One of the joys of getting old is that you forget things. So one of the things I recall is that the converse of the Pythagorean Theorem is true, that is, if a triangle with short sides a and b and long side c is such that

$$a^2 + b^2 = c^2,$$

then the triangle must be a right triangle with the angle between sides a and b being 90°. But I didn't recall how to prove it. So I thought I would see if I could do it without looking up any sources.

Solution

I made some attempts at proving the converse using plane geometry, but couldn't see an easy way to do it. Even though there are over a hundred proofs of the original Pythagorean Theorem, they are not that easy to think of without knowing them first.

So I tried analytic geometry with better results.

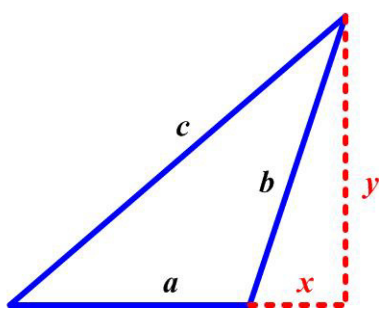


Figure 1

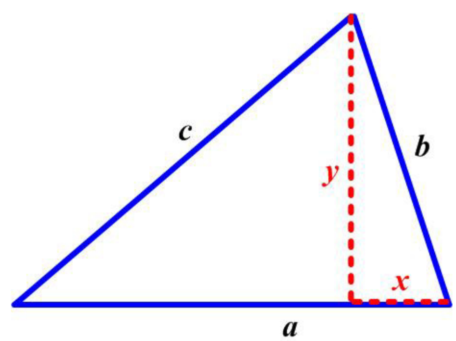


Figure 2

Figure 1 shows the case when the angle in question is obtuse and Figure 2 when the angle is acute. A perpendicular of length y is dropped from the end of line c and lands a (signed) distance x from the end of line a. So besides the given relationship $a^2 + b^2 = c^2$, we have from the original Pythagorean Theorem

$$x^2 + y^2 = b^2 \quad \text{and} \quad (a + x)^2 + y^2 = c^2 \qquad (*)$$

where x may be positive or negative.

Therefore $$a^2 + 2ax + x^2 + y^2 = c^2 = a^2 + b^2$$

or $$2ax + b^2 = b^2$$

or $$2ax = 0 \;\Rightarrow\; x = 0 \text{ and } y = b.$$

And that means a is perpendicular to b and we have a right angle.

Comment 1. A slightly different way of looking at the situation is to just begin with equations (*) and not assume $a^2 + b^2 = c^2$. So

$$a^2 + 2ax + x^2 + y^2 = c^2$$

or

$$a^2 + 2ax + b^2 = c^2.$$

And so

$$c^2 - (a^2 + b^2) = 2ax > 0 \text{ if } x > 0 \text{ (obtuse)}$$

$$c^2 - (a^2 + b^2) = 2ax < 0 \text{ if } x < 0 \text{ (acute).}$$

That is, if the angle is not a right angle, then $a^2 + b^2 \neq c^2$, which is the other way of proving the converse.

Comment 2. This is sort of subliminal but I was following Polya's principle of reducing the problem to one I already knew how to solve, namely, the original Pythagorean Theorem. Then I just followed the computations, hoping something nice would happen, and it did. A finished solution always gives the impression you knew everything beforehand, but that is not the case, which is why solving problems is so frustrating and rewarding.