Special Log Sum

11 January 2024

Jim Stevenson

Here is a fairly computationally challenging 1994 AIME problem ([1]).

Find the positive integer n for which

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \ldots + \lfloor \log_2 n \rfloor = 1994.$$

where for real x, $\lfloor x \rfloor$ is the greatest integer $\leq x$.

There is some fussy consideration of indices.

Solution

Let $f(n) = \lfloor \log_2 n \rfloor$, so $f(2^k) = k$, and $f(n) = k$ for $2^k \leq n < 2^{k+1}$. And let

$$S(n) = f(1) + f(2) + \ldots + f(n).$$

Consider some initial values and the corresponding behavior:

n	1	2	3	2^2	5	6	7	2^3	9	10	11	12	13	14	15	2^4	…	2^k-1	2^k	…
$f(n)$	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3	4	…	k -1	k	…
$S(n)$	0	+ 2·1					+ 2^2·2								+ 2^3·3	+4	…	+ $2^{k-1}(k-1)$	$+k$	…

Then

$$S(n) = T(k) + (n - 2^k + 1)k \quad \text{for } 2^k \leq n < 2^{k+1},$$

where

$$T(k) = \sum_{m=1}^{k-1} m2^m$$

We want to find n such that $S(n) = 1994$, so we want to find the largest k such that $T(k) \leq 1994$. We will consider some alternative approaches, but a brute-force computation gives:

k	2	3	4	5	6	7	8	9
$(k-1)2^{k-1}$	1·2 = 2	2·4 = 8	3·8 = 24	4·16 = 64	5·32 = 160	6·64 = 384	7·128 = 896	8·256 = 2048
$T(k)$	2	10	34	98	258	642	1538	3586

So the largest k is 8. So

$$1994 - T(8) = 1994 - 1538 = 456 = (n - 2^8 + 1)8 = (n - 255)8$$

and so

$$n = 255 + 57 = 312$$

(This is essentially the same solution as given by AIME, only with a bit more detail.)

Notice that this problem would work for the years 2002, 2010, 2018, and 2026, that is, multiples of 8 beyond 1994. So 312 + 4 = 316 would give the value of n for 2026.

Alternative Calculation for $T(k) = \sum_{m=1}^{k-1} m2^m$

This sequence looks familiar. We encountered its infinite form in the "Amazing Root Problem"[1] and "Another Challenging Sum".[2] Apparently it is called the *arithmetico-geometric series*. Wikipedia derives an expression for its partial sums,[3] but I thought it would be interesting to try to apply my standard geometric series approach. It does involve a bit of computation that actually makes the brute-force method a bit faster.

As before, let $G_k(x)$ be the kth partial sum of the geometric series

$$G_k(x) = 1 + x + x^2 + x^3 + \ldots + x^k = \frac{1 - x^{k+1}}{1 - x}$$

Then

$$G_k'(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots + kx^{k-1} = \frac{(x-1)(k+1)x^k + (1 - x^{k+1})}{(1-x)^2}$$

So

$$T(k) = 2G_{k-1}'(2) = 2 + (k - 2)2^k$$

and

$$T(8) = 2 + 6 \cdot 2^8 = 2 + 1536 = 1538$$

as we got before.

References

[1] "Problem 4" 1994 AIME Problems (https://artofproblemsolving.com/wiki/index.php/1994_AIME_Problems)

[1] https://josmfs.net/2023/12/30/amazing-root-problem/
[2] https://josmfs.net/2023/12/09/another-challenging-sum/
[3] https://en.wikipedia.org/wiki/Arithmetico-geometric_sequence