Distance to Flag Problem

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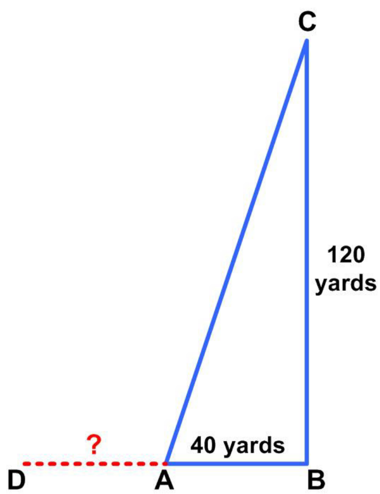

The following puzzle is from the Irishman Owen O'Shea ([1]).

The figure shows the location of three flags [at A, B, and C] in one of the fields on a neighbor's farm. The angle ABC is a right angle. Flag A is 40 yards from Flag B. Flag B is 120 yards from flag C. Thus, if one was to walk from A to B and then on to C, one would walk a total of 160 yards.

Now there is a point, marked by flag D, [directly] to the left of flag A. Curiously, if one were to walk from flag A to flag D and then diagonally across to flag C, one would walk a total distance of 160 yards.

The question for our puzzlers is this: how far is it from flag D to flag A?

This problem has a simple solution. But it also suggests a more advance alternative approach.

Solution 1

As shown in Figure 1, let x be the distance from flag A to flag D and d the distance from flag D to flag C. We are given that

$$x + d = 160 \text{ yards.}$$

Since the triangle is a right triangle, we have by the Pythagorean Theorem that

$$d^2 = (x + 40)^2 + 120^2$$

Now substituting the first relationship into the second equation yields

$$d^2 = (160 - d + 40)^2 + 120^2 = (200 - d)^2 + 120^2 = 200^2 - 400\text{d} + d^2 + 120^2$$

So canceling the d^2 and simplifying, we get

$$40{\cdot}10d = (5{\cdot}40)^2 + (3{\cdot}40)^2 = 34{\cdot}40^2$$

or

$$d = 136.$$

Therefore

$$x = 160 - 136 = 24 \text{ yards}.$$

(O'Shea's solution is virtually the same.)

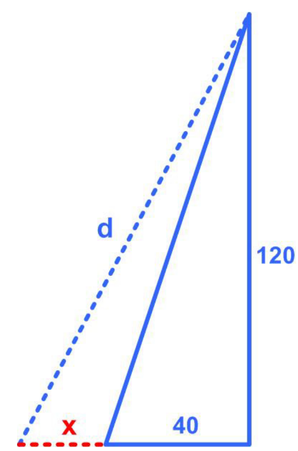

Figure 1

Solution 2

Seeing that the puzzle involves two sums that are the same, namely 160 yards, immediately suggests that an ellipse might be lurking somewhere. Figure 2 shows that there is. Flags at A and C are positioned at the foci of the ellipse, which means the center is half the distance between these points, or (via the Pythagorean Theorem) the distance from focus to center $c = 40\sqrt{10} / 2 = 20\sqrt{10}$. The constant distance 160 represents twice the length of the semi-major axis a, so that $a = 160/2 = 80$.

Now we use the parametric equation for the ellipse with origin at one focus as given in the post "Kepler's Laws and Newton's Laws",[1] namely,

$$r = \frac{p}{1 + e\cos\theta}$$

where eccentricity

$$e = c/a = \sqrt{10}/4,$$

and

$$p = a(1 - e^2) = 80 \cdot 6/16 = 30.$$

Therefore,

$$r = \frac{30}{1 + \frac{\sqrt{10}}{4}\cos\theta}$$

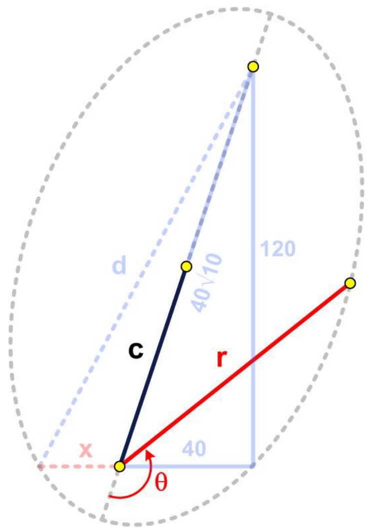

Figure 2

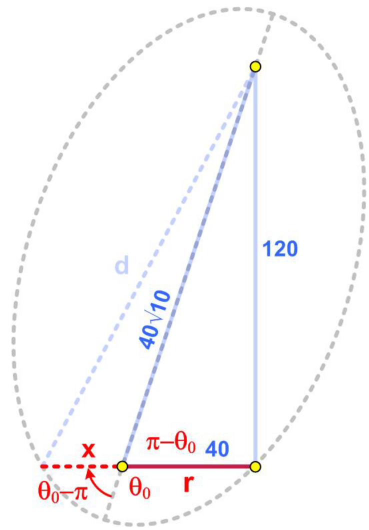

Figure 3

Let $\theta = \theta_0$ when r is horizontal (and equals 40) (Figure 3). Then the desired distance x is when r has rotated $\theta_0 - \pi$. So

$$x = \frac{30}{1 + \frac{\sqrt{10}}{4}\cos(\theta_0 - \pi)}$$

But $\cos(\theta_0 - \pi) = \cos(\pi - \theta_0) = 40 / 40\sqrt{10} = 1/\sqrt{10}$. Therefore,

$$\boxed{x = \frac{30}{1 + \frac{\sqrt{10}}{4}\frac{1}{\sqrt{10}}} = 24}$$

which agrees with our previous answer. Cool, if a bit over-kill.

References

[1] O'Shea, Owen, *Mathematical Brainteasers with Surprising Solutions*, Prometheus Books, Guilford, Connecticut, 2020

[1] https://josmfs.net/2018/12/29/keplers-laws-and-newtons-laws/