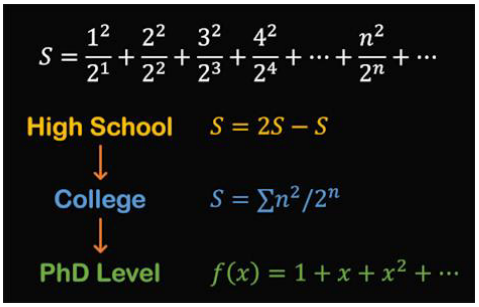
Another Challenging Sum

8 December 2023

Jim Stevenson



This is yet another series offered by Presh Talwalkar.[1]

What is the value of the following sum?

$$S = \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots + \frac{n^2}{2^n} + \cdots$$

Talwalkar gives hints for three possible approaches to the solution.

My Solution

I proceed as in previous cases,[2] namely, via power series. Let

$$H(x) = \sum_{n=1}^{\infty} n^2 x^n$$

Then $H(½) = S$. Now $H(x)$ suggests differentiation of the geometric power series (where we can differentiate converging power series term-by-term just as if they were polynomials. This is why Newton used them so much in his work.). Let $F(x)$ be the geometric series

$$F(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \qquad |x| < 1$$

Then
$$F'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}$$

and
$$F''(x) = \frac{1}{(1-x)^3}$$

Let
$$G(x) = xF'(x) = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \ldots = \frac{x}{(1-x)^2}$$

Then
$$G'(x) = xF''(x) + F'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \ldots = \frac{x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

and
$$H(x) = xG'(x) = \sum_{n=1}^{\infty} n^2 x^n = x^2 F''(x) + xF'(x) = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}$$

so
$$\boxed{S = H(\tfrac{1}{2}) = \frac{\frac{1}{2} \cdot \frac{3}{2}}{(\frac{1}{2})^3} = 6}$$

[1] https://mindyourdecisions.com/blog/2023/12/04/sum-of-n-squared-over-2-to-n/

[2] Such as, "Autumn Sum" (https://josmfs.net/2020/10/24/autumn-sum/) and "Winter Sum" (https://josmfs.net/2022/01/15/winter-sum/).

Talwalkar Solutions

First we will check for convergence.

$$S_n = \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots + \frac{n^2}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{1}{2}\cdot\left(\frac{n+1}{n}\right)^2\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{2}\cdot\left(1+\frac{1}{n}\right)^2\right| = \lim_{n\to\infty}\left|\frac{1}{2}\cdot 1^2\right| = \frac{1}{2} < 1$$

Now let's solve it in a few ways.

Method 1: pattern

$$S = 1^2/2^1 + 2^2/2^2 + 3^2/2^3 + \ldots + n^2/2^n + \ldots$$

$$S = 1/2 + 4/4 + 9/8 + 16/16 + 25/32 + \ldots$$

$$2S = 1 + 4/2 + 9/4 + 16/8 + 25/16 + \ldots$$

$$2S - S = S = 1 + 3/2 + 5/4 + 7/8 + 9/16 + \ldots$$

$$S - 1 = 3/2 + 5/4 + 7/8 + 9/16 + \ldots$$

$$2(S - 1) = 3 + 5/2 + 7/4 + 9/8 + \ldots$$

$$2(S - 1) - S = 2 + 2/2 + 2/4 + 2/8 + \ldots$$

Simplifying both sides we get:

$$S - 2 = 2 + 1 + 1/2 + 1/4 + \ldots$$

$$S - 2 = 2 + 1/(1\text{-}1/2)$$

$$S - 2 = 2 + 2$$

So we have: $S = 6$

Method 2: sequences

$$S_n = \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots + \frac{n^2}{2^n}$$

$$S = \sum_{n=1}^{\infty}\frac{n^2}{2^n} = \sum_{n=0}^{\infty}\frac{(n+1)^2}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{n^2}{2^n}$$

$$S = 2S - S = 2\sum_{n=0}^{\infty}\frac{(n+1)^2}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{n^2}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{n^2 + 2n + 1 - n^2}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{2n}{2^n} + \frac{1}{2^n}$$

$$= 2\sum_{n=0}^{\infty} \frac{n}{2^n} + \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= 2 \times 2 + 2 = 6$$

The sequence $n/2^n$ is an arithmetico-geometric series[3] and the series $1/2^n$ is a standard geometric series.

(Wikipedia) The summation of this infinite sequence is known as an **arithmetico-geometric series**, and its most basic form has been called **Gabriel's staircase**:

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}, \quad \text{for } 0 < r < 1$$

I prefer derivations to memorization of formulas, so I would derive this result directly. In fact, note that this is the same as my $G(x)$ above, that is, $G(r) = r\,F'(r) = r \,/\, (1 - r)^2$. So $G(½) = 2$.

Method 3: generating function

This is the same as my solution.

References

Answers by Trevor, Aryan Arora, Alexey Godin

https://www.quora.com/How-do-you-evaluate-the-sum-of-n-2-2-n-from-n-1-to-infinity

Convergence

https://socratic.org/questions/how-do-you-test-the-series-sigma-n-2-2-n-from-n-is-0-oo-for-convergence

[3] https://en.wikipedia.org/wiki/Arithmetico-geometric_sequence