Amazing Root Problem

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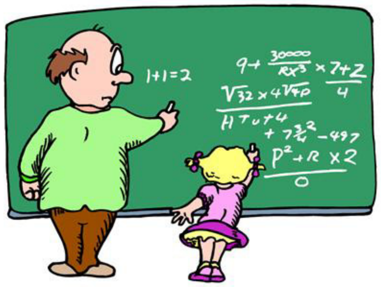

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This is a challenging but imaginative problem from the 2024 Math Calendar ([1]).

$$\sqrt{2\sqrt{4\sqrt{8\sqrt{16\sqrt{\ldots}}}}}$$

As before, recall that all the answers are integer days of the month.

Solution

The solution came to me incrementally. First I noticed the √2 could be pulled out:

$$\sqrt{2}\left(4\sqrt{8\sqrt{16\sqrt{\ldots}}}\right)^{\frac{1}{4}}$$

And then the ¼ th root of 4:

$$2^{\frac{1}{2}}4^{\frac{1}{4}}\left(8\sqrt{16\sqrt{\ldots}}\right)^{\frac{1}{8}}$$

And then the pattern became clear:

$$2^{\frac{1}{2}}4^{\frac{1}{2^2}}8^{\frac{1}{2^3}}\ldots = 2^{\frac{1}{2}}2^{\frac{2}{2^2}}2^{\frac{3}{2^3}}\ldots = 2^S$$

where S is the sum

$$S = \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots + \frac{n}{2^n} + \ldots$$

We have seen this sum before (in the solution to “Another Challenging Sum”[1]) or we can apply the usual trick of considering the power series

$$S(x) = \sum_{n=1}^{\infty} nx^n$$

where $S = S(½)$.

Again we use the geometric series:

$$G(x) = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

Taking the derivative,

$$G'(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots = \frac{1}{(1-x)^2}$$

[1] https://josmfs.net/2023/12/09/another-challenging-sum/

Then

$$S(x) = xG'(x) = \frac{x}{(1-x)^2}$$

Evaluating at $x = 1/2$,

$$S = S(\tfrac{1}{2}) = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

So the answer to the problem is

$$\boxed{\sqrt{2\sqrt{4\sqrt{8\sqrt{16\sqrt{\ldots}}}}} = 2^2 = 4}$$

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2024: Your Daily epsilon of Math*, American Mathematical Society, 2024. January