Elliptic Circles

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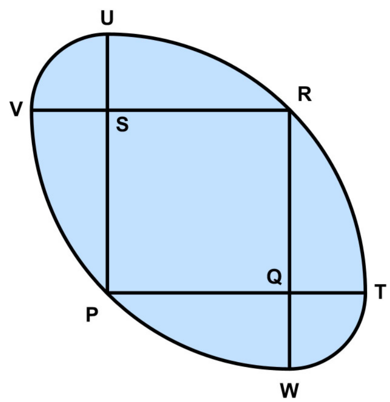
Here is another UKMT Senior Challenge problem for 2017.



The diagram shows a square PQRS with edges of length 1, and four arcs, each of which is a quarter of a circle. Arc TRU has centre P; arc VPW has centre R; arc UV has centre S; and arc WT has centre Q.

What is the length of the perimeter of the shaded region?

A 6 B $(2\sqrt{2} - 1)\pi$ C $(\sqrt{2} - {}^{1}/_{2})\pi$ D 2π E $(3\sqrt{2} - 2)\pi$

Solution

Actually, the solution is rather straight-forward. Figure 1 shows the values given in the problem and those derived from the given values. Using the fact that the arclength of an arc of a circle is the radius times the angle subtended by the arc in radians, we have

$$\begin{aligned}\text{Perimeter} &= 2\ \text{UT} + 2\text{UV}\\ &= 2\sqrt{2}\ \pi/2 + 2(\sqrt{2} - 1)\ \pi/2\\ &= (2\sqrt{2} - 1)\ \pi\end{aligned}$$

or answer B.

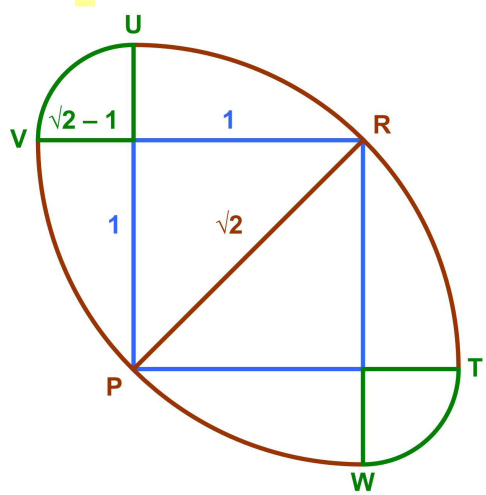


Figure 1 Problem Solution

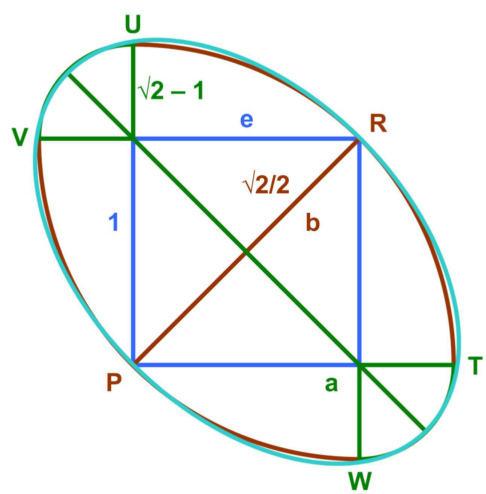


Figure 2 Ellipse Approximation

Comment. Clearly the figure looks like an ellipse, so I was wondering how close an approximation to an ellipse it was. Figure 2 shows the answer. The ellipse has semimajor axis $a = 3\sqrt{2}/2 - 1$ and semiminor axis $b = \sqrt{2}/2$.

The two circular arcs are from circles that approximate the osculating circles giving the maximum and minimum radii of curvature of the ellipse. How close is this approximation? From Wikipedia, the radius of curvature of the ellipse at the semimajor axis vertex is $R_a = b^2/a$ and at the semiminor axis vertex is $R_b = a^2/b$.

Substituting our values we get the radius of curvature at the semimajor axis vertex is

$$R_a = b^2/a = 1/(3\sqrt{2} - 2) \approx 0.446$$

and the radius of the circular arc UV containing the semimajor axis vertex is

$$\sqrt{2} - 1 \approx 0.414$$

At the semiminor axis vertex we get the radius of curvature is

$$R_b = a^2/b = (11 - 6\sqrt{2})/\sqrt{2} \approx 1.778$$

and the radius of the arc UT containing the semiminor axis vertex is

$$\sqrt{2} \approx 1.414$$

These results agree with the ellipse in Figure 2 whose radii of curvature at each of these vertices are larger than those of the approximating circular arcs.