Maximized Box Problem

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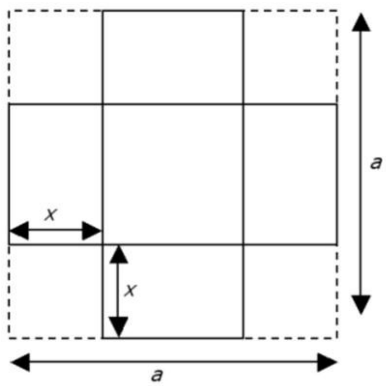
31 August 2023



This problem is from Colin Hughes's *Maths Challenge* website (mathschallenge.net) ([1]).

Four corners measuring x by x are removed from a sheet of material that measures a by a to make a square based open-top box. Prove that the volume of the box is maximised iff the area of the base is equal to the area of the four sides.

My Solution

From Figure 1 the volume of the box is given by

$$V(x) = x(a - 2x)^2,$$

the area of the base by

$$A_b = (a - 2x)^2.$$

and the area of the sides by

$$A_s = 4x(a - 2x),$$

where $0 \le x \le a/2$ (Figure 1). Note that $V(0) = V(a/2) = 0$ and $V(x) > 0$ for $0 < x < a/2$. Since $V(x)$ is continuous on the closed interval $[0, a/2]$ and not identically zero, there is at least one point in the open interval $(0, a/2)$ where it achieves its maximum. Moreover, $V(x)$ being a polynomial, it is continuously differentiable and its derivative must vanish at that maximum point.

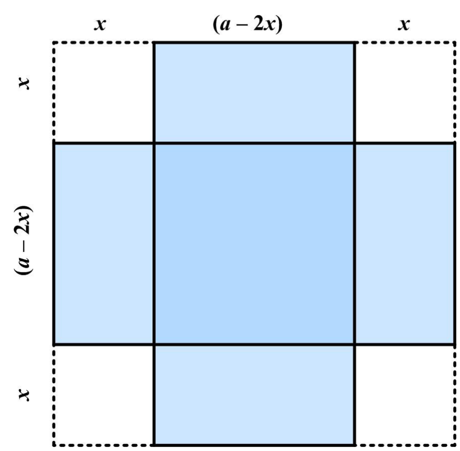


Figure 1

Now

$$0 = dV/dx = x(a - 2x)(-2) + (a - 2x)^2 \iff$$

$$(a - 2x)^2 = 4x(a - 2x) \iff$$

$$A_b = A_s$$

We could solve for x and show there is only one solution and therefore the maximum, or test the second derivative to verify that it is a maximum. Alternatively, we could realize $V(x)$ describes a cubic polynomial with a zero at $x = 0$ and a double zero at $x = a/2$. Extending the range of x into negative values we see $V(x)$ becomes negative and for all $x > 0$, $V(x) \ge 0$. So $V(x)$ takes the form shown in Figure 2. And therefore it has only one critical point inside the interval $(0, a/2)$, and it must be a maximum. Moreover, it is a critical point ($V'(x) = 0$) iff $A_b = A_s$.

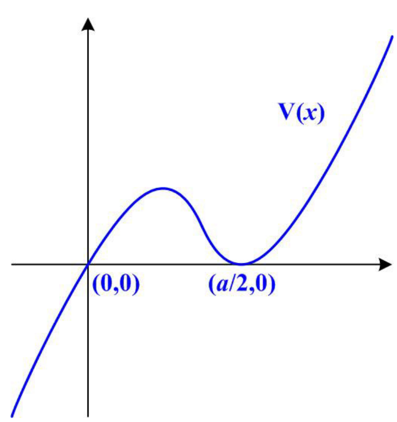
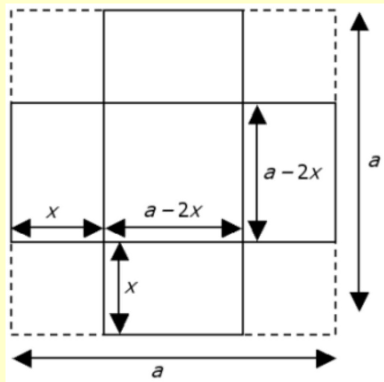


Figure 2

Maths Challenge Solution

The Maths Challenge solution is basically the same as mine.

Consider the diagram.



$$A_{base} = (a - 2x)^2 = a^2 - 4ax + 4x^2$$

∴Volume of box, $V = a^2x - 4ax^2 + 4x^2$

$$dV/dx = a^2\ 8ax + 12x^2$$

At turning point, dV/dx = 0

$$\therefore\ a^2 - 8ax + 12x^2 = 0$$

$$(a - 2x)(a - 6x) = 0$$

$$\therefore\ x = a/2,\ a/6.$$

Clearly $x = a/2$ is a trivial solution, as $A_{base} = 0$, so we need only consider the solution $x = a/6$.

$$d^2V/dx^2 = -8a + 24x.$$

When $x = a/6$, $d^2V/dx^2 = -4a < 0 \Rightarrow$ V is at a maximum value.

Note that it is not sufficient to show that the area of the base equals the area of the sides when $x = a/6$, as we are attempting to prove that the volume is maximised if and only if this condition is true.

$$A_{sides} = 4x(a - 2x)$$

Solving $A_{base} = A_{sides}$, $(a - 2x)^2 = 4x(a - 2x)$

$$\therefore\ (a - 2x)^2 - 4x(a - 2x) = 0$$

$$(a - 2x)((a - 2x) - 4x) = 0$$

$$(a - 2x)(a - 6x) = 0$$

$$\therefore\ x = a/2,\ a/6.$$

We reject $x = a/2$, as $A_{base} = 0$.

Hence $A_{base} = A_{sides}$ has a unique non-trivial solution, $x = a/6$, which is when the volume of the box is maximised.

References

[1] Hughes, Colin, “Maximised Box”, *Maths Challenge*, (mathschallenge.net) (May 2003) #122 p.73. Difficulty: 4 Star. “A four-star problem: A comprehensive knowledge of school mathematics and advanced mathematical tools will be required.”