Amazing Identity

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This is a most surprising and amazing identity from the 1965 Polish Mathematical Olympiads ([1]).

31. Prove that if n is a natural number, then we have

$$(\sqrt{2} - 1)^n = \sqrt{m} - \sqrt{(m - 1)},$$

where m is a natural number.

Here, natural numbers are 1, 2, 3, …

I found it to be quite challenging, as all the Polish Math Olympiad problems seem to be.

My Solution

We will proceed by mathematical induction, but it is a bit complicated. As mentioned in earlier posts, for example "Surprising Identity",[1] the basic principle is as follows:

Principle of Mathematical Induction:

Given: (i) Statement P(n) associated with each natural number $n = 1, 2, 3, \ldots$

(ii) P(1) is true.

(iii) For all natural numbers $k = 1, 2, 3, \ldots$, if P(k) is true, then P(k+1) is true.

Then: For all natural numbers $n = 1, 2, 3, \ldots$, P(n) is true.

Provisionally, we can define P(n): $(\sqrt{2} - 1)^n = \sqrt{m} - \sqrt{(m - 1)}$, for some natural number m. We wish to get a more refined expression for m. To get an idea how that might work in this case we look at some initial statements.

When $n = 1$, we have

$$(\sqrt{2} - 1)^1 = \sqrt{2} - \sqrt{1}$$

and so the pattern holds trivially. For $n = 2$, we have

$$(\sqrt{2} - 1)^2 = (\sqrt{2} - 1)\,(\sqrt{2} - 1) = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2} = \sqrt{9} - \sqrt{8}$$

and so the pattern holds. Finally, consider $n = 3$.

$$(\sqrt{2} - 1)^3 = (\sqrt{2} - 1)^2\,(\sqrt{2} - 1) = (3 - 2\sqrt{2})\,(\sqrt{2} - 1) = 5\sqrt{2} - 7 = \sqrt{50} - \sqrt{49}.$$

Again the statement P(n): $(\sqrt{2} - 1)^n = \sqrt{m} - \sqrt{(m - 1)}$ holds for $n = 3$.

In the process we notice another pattern, namely

$$(a - b\sqrt{2})\,(\sqrt{2} - 1) = a'\sqrt{2} - b' \quad\text{and}\quad (a\sqrt{2} - b)\,(\sqrt{2} - 1) = a' - b'\sqrt{2}$$

The position of $\sqrt{2}$ oscillates between the first term and the second term on each iteration. Let's codify this. P(n) is $(\sqrt{2} - 1)^n = a_n\sqrt{2} - b_n$ when n is odd (because it satisfies that pattern for $n = 1$) and P(n) is $(\sqrt{2} - 1)^n = a_n - b_n\sqrt{2}$ when n is even (because it satisfies that pattern for $n = 2$).

[1] https://josmfs.net/2021/06/26/surprising-identity/

Now we consider the "induction step" (iii).

Assume *k* is even and P(k) is true. Then

$$P(k) \text{ is } (\sqrt{2} - 1)^k = a_k - b_k\sqrt{2} = \sqrt{a_k^2} - \sqrt{(2b_k^2)}$$

where $a_k^2 - 1 = 2b_k^2$.

Now $(\sqrt{2} - 1)^{k+1} = (\sqrt{2} - 1)^k (\sqrt{2} - 1) = (a_k - b_k\sqrt{2})(\sqrt{2} - 1) = (a_k + b_k)\sqrt{2} - (a_k + 2b_k)$

so define $$a_{k+1} = a_k + b_k \text{ and } b_{k+1} = a_k + 2b_k$$

Then $$(\sqrt{2} - 1)^{k+1} = a_{k+1}\sqrt{2} - b_{k+1} = \sqrt{(2\,a_{k+1}^2)} - \sqrt{b_{k+1}}^2$$

Now $$2a_{k+1}^2 - 1 = b_{k+1}^2$$

if and only if (iff) $$2(a_k + b_k)^2 - 1 = (a_k + 2b_k)^2$$

iff $$2(a_k^2 + 2\,a_k\,b_k + b_k^2) - 1 = a_k^2 + 4\,a_k\,b_k + 4b_k^2$$

iff $$2a_k^2 + 4\,a_k\,b_k + 2b_k^2 - 1 = a_k^2 + 4\,a_k\,b_k + 4b_k^2$$

iff $$a_k^2 - 1 = 2b_k^2$$

which is true from P(k). Therefore P(k) $\Rightarrow$ P(k+1) when k is even.

Assume *k* is odd and P(k) is true. Then

$$P(k) \text{ is } (\sqrt{2} - 1)^k = \sqrt{2}\,a_k - b_k = \sqrt{(2a_k^2)} - b_k^2$$

where $2a_k^2 - 1 = b_k^2$.

Now $(\sqrt{2} - 1)^{k+1} = (\sqrt{2} - 1)^k (\sqrt{2} - 1) = (\sqrt{2}a_k - b_k)(\sqrt{2} - 1) = (2a_k + b_k) - (a_k + b_k)\sqrt{2}$

so define $$a_{k+1} = 2a_k + b_k \text{ and } b_{k+1} = a_k + b_k$$

Then $$(\sqrt{2} - 1)^{k+1} = a_{k+1} - b_{k+1}\sqrt{2} = \sqrt{a_{k+1}}^2 - \sqrt{(2b_{k+1}^2)}$$

Now $$a_{k+1}^2 - 1 = 2b_{k+1}^2$$

iff $$(2a_k + b_k)^2 - 1 = 2(a_k + b_k)^2$$

iff $$4a_k^2 + 4\,a_k\,b_k + b_k^2 - 1 = 2(a_k^2 + 2\,a_k\,b_k + b_k^2)$$

iff $$4a_k^2 + 4\,a_k\,b_k + b_k^2 - 1 = 2a_k^2 + 4\,a_k\,b_k + 2b_k^2$$

iff $$2a_k^2 - 1 = b_k^2$$

which is true from P(k). Therefore P(k) $\Rightarrow$ P(k+1) when k is odd.

Therefore, we have shown P(1) is true, and for all k = 1, 2, 3, … P(k) $\Rightarrow$ P(k+1) is true. Therefore, for all natural numbers n, there is a natural number m, such that

$$P(n): (\sqrt{2} - 1)^n = \sqrt{m} - \sqrt{(m - 1)}$$

is true.

Olympiad Solution

Again I present their solutions as images below (p.4). It turns out their Method I is what I thought of, only with the addition of a neat observation that eliminated the even/odd separation of cases. In skimming their solutions, there seemed to be a question whether some of the operations resulted in a natural number. It didn't seem that I had to worry about that in my solution.

References

[1] Straszewicz, S., *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads*, J. Smolska, tr., Popular Lectures in Mathematics, Vol.12, Pergamon Press, London, 1965 (Polish edition 1960). Problem p.5, solution p.43.

Olympiad Solution

31. *Method I.* First, we shall prove that for any natural n there exist natural numbers a and b such that

$$(1-\sqrt{2})^n = \sqrt{a^2}-\sqrt{(2b^2)},$$

and

$$a^2-2b^2 = (-1)^n.$$

Proof. For $n=1$ the theorem is valid, namely $a=b=1$. Suppose that the theorem is valid for a certain n; then

$$\begin{aligned}(1-\sqrt{2})^{n+1} &= (1-\sqrt{2})^n(1-\sqrt{2}) = [\sqrt{a^2}-\sqrt{(2b^2)}](1-\sqrt{2})\\ &= (a-b\sqrt{2})(1-\sqrt{2}) = (a+2b)-(a+b)\sqrt{2}\\ &= \sqrt{[(a+2b)^2]}-\sqrt{[2(a+b)^2]}\\ &= \sqrt{a_1^2}-\sqrt{(2b_1^2)},\end{aligned}$$

where a_1 and b_1 are natural numbers and

$$\begin{aligned}a_1^2-2b_1^2 &= (a+2b)^2-2(a+b)^2 = -a^2+2b^2\\ &= -(a^2-2b^2) = (-1)^{n+1}.\end{aligned}$$

Thus the theorem is also valid for the exponent $n+1$. Hence we infer by induction that the theorem is valid for any natural n.

The theorem involved in the problem is an immediate conclusion from the theorem proved above; for, if n is an even number, then

$$(\sqrt{2}-1)^n = (1-\sqrt{2})^n = \sqrt{a^2}-\sqrt{(2b^2)},$$

where a and b, and therefore also a^2 and $2b^2$, are natural numbers and $a^2-2b^2=1$. If n is an odd number, then

$$(\sqrt{2}-1)^n = -(1-\sqrt{2})^n = \sqrt{(2b^2)}-\sqrt{a^2},$$

where $2b^2$ and a^2 are natural numbers and

$$2b^2-a^2 = -(a^2-2b^2) = -(-1) = 1.$$

Method II. Since

$$(\sqrt{2}-1)^n = \frac{(\sqrt{2}+1)^n+(\sqrt{2}-1)^2}{2} - \frac{(\sqrt{2}+1)^n-(\sqrt{2}-1)^n}{2},$$

we have

$$(\sqrt{2}-1)^n = \sqrt{m}-\sqrt{k},$$

where

$$m = \left[\frac{(\sqrt{2}+1)^n+(\sqrt{2}-1)^n}{2}\right]^2 = \frac{(\sqrt{2}+1)^{2n}+(\sqrt{2}-1)^{2n}+2}{4},$$

$$k = \left[\frac{(\sqrt{2}+1)^n-(\sqrt{2}-1)^n}{2}\right]^2 = \frac{(\sqrt{2}+1)^{2n}+(\sqrt{2}-1)^{2n}-2}{4},$$

and consequently

$$m-k=1, \quad \text{i.e.} \quad k=m-1.$$

It remains to prove that m is a natural number. According to Newton's binomial formula we have

$$(\sqrt{2}+1)^n = (\sqrt{2})^n+\binom{n}{1}(\sqrt{2})^{n-1}+\binom{n}{2}(\sqrt{2})^{n-2}+\binom{n}{3}(\sqrt{2})^{n-3}+\ldots$$

$$(\sqrt{2}-1)^n = (\sqrt{2})^n-\binom{n}{1}(\sqrt{2})^{n-1}+\binom{n}{2}(\sqrt{2})^{n-2}-\binom{n}{3}(\sqrt{2})^{n-3}+\ldots,$$

and thus

$$\sqrt{m} = \frac{(\sqrt{2}+1)^n+(\sqrt{2}-1)^n}{2} = (\sqrt{2})^n+\binom{n}{2}(\sqrt{2})^{n-2}+\ldots$$

The above equality implies that, if n is even, $\sqrt{m}$ is a sum of natural numbers, whence $\sqrt{m}$ and m are natural numbers. If n is odd, then $\sqrt{m}$ is a sum of numbers of the form $a\sqrt{2}$, where a is a natural number; consequently $\sqrt{m}$ is the product of a natural number and $\sqrt{2}$, whence m is a natural number. The theorem is thus proved.

We shall give two more solutions of the problem, not so concise as the above two but having the advantage of suggesting themselves quite naturally.

Method III. We shall use the induction method. The theorem stating that for a natural n we have the equality

$$(\sqrt{2}-1)^n = m-\sqrt{(m-1)} \quad (m\text{—natural number})$$

is valid if $n=1$; in this case $m=2$. Suppose that it is valid for a certain natural n; then

$$\begin{aligned}(\sqrt{2}-1)^{n+1} &= (\sqrt{2}-1)^n(\sqrt{2}-1) = [(\sqrt{m}-\sqrt{(m-1)}](\sqrt{2}-1)\\ &= \sqrt{(2m)}+\sqrt{(m-1)}-\sqrt{[2(m-1)]}-\sqrt{m}\\ &= \sqrt{[\sqrt{(2m)}+\sqrt{(m-1)}]^2}-\sqrt{\{\sqrt{[2(m-1)]}+\sqrt{m}\}^2},\end{aligned}$$

and

$$\begin{aligned}&[\sqrt{(2m)}+\sqrt{(m-1)}]^2-\{\sqrt{[2(m-1)]}+\sqrt{m}\}^2\\ &= \{3m-1+2\sqrt{[2m(m-1)]}\}-\{3m-2+2\sqrt{[2m(m-1)]}\} = 1.\end{aligned}$$

We shall prove that $[\sqrt{2m}+\sqrt{(m-1)}]^2$ is a natural number. Since

$$[\sqrt{(2m)}+\sqrt{(m-1)}]^2 = 3m-1+2\sqrt{[2m(m-1)]},$$

it is sufficient to prove that $2m(m-1)$ is a square of a natural number. It will be observed that $(\sqrt{2}-1)^n$ is a number of the form $a\sqrt{2}+b$, where a and b are integers, since each term of the expansion of $(\sqrt{2}-1)^n$ according to Newton's formula is either an integer or the product of an integer and $\sqrt{2}$. Consequently, by the induction hypothesis

$$\sqrt{m}-\sqrt{(m-1)} = a\sqrt{2}+b.$$

By squaring, we obtain

$$2m-1-2\sqrt{[m(m-1)]} = 2a^2+b^2+2ab\sqrt{2}.$$

It follows that

$$-2\sqrt{[m(m-1)]} = 2ab\sqrt{2}, \quad 2m(m-1) = 4a^2b^2,$$

and thus the number $2m(m-1)$ is the square of the natural number $2|ab|$.

We have shown that the theorem is valid for the exponent $n+1$ if it is valid for the exponent n. And since it is valid for $n = 1$, it is valid for any natural n.

Remark. In the end part of the above proof we assumed the following theorem:

If

$$A+B\sqrt{C} = K+L\sqrt{M},$$

where A, B, C, K, L, M are rational numbers, $L \neq 0$, and M is not a square of a rational number, then

$$A = K \quad \text{and} \quad B\sqrt{C} = L\sqrt{M}.$$

The proof of this theorem is simple. The equality assumed implies that

$$B\sqrt{C} = K-A+L\sqrt{M}.$$

Hence

$$B^2C = (K-A)^2+2(K-A)L\sqrt{M}+L^2M,$$

$$2(K-A)L\sqrt{M} = B^2C-L^2M-(K-A)^2.$$

The right-hand side of the last equation represents a rational number, and thus the left-hand side must also be equal to a rational number; under the assumption made regarding L and M, this occurs only for $K = A$.

Method IV is actually a slight modification of method II. The equality

$$(\sqrt{2}-1)^n = \sqrt{m}-\sqrt{(m-1)} \tag{1}$$

is regarded as an equation with the unknown m. We solve this equation:

$$\sqrt{m} = (\sqrt{2}-1)^n+\sqrt{(m-1)},$$

$$m = (\sqrt{2}-1)^{2n}+2(\sqrt{2}-1)^n\sqrt{(m-1)}+m-1,$$

$$\sqrt{(m-1)} = \frac{1-(\sqrt{2}-1)^{2n}}{2(\sqrt{2}-1)^n} = \frac{1-(\sqrt{2}-1)^{2n}}{2(\sqrt{2}-1)^n} \times \frac{(\sqrt{2}+1)^n}{(\sqrt{2}+1)^n}$$

$$= \frac{1}{2}[(\sqrt{2}+1)^n-(\sqrt{2}-1)^n],$$

$$m-1 = \frac{1}{4}[(\sqrt{2}+1)^n-(\sqrt{2}-1)^n]^2,$$

$$m = \frac{1}{4}[(\sqrt{2}+1)^n-(\sqrt{2}-1^n)]^2+1$$

$$= \frac{1}{4}[(\sqrt{2}+1)^{2n}+(\sqrt{2}-1)^{2n}-2]+1$$

$$= \frac{1}{4}[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2.$$

Substituting in equation (1) this value of m we find that it satisfies that equation. The proof that it is a natural number has been given in method II.