String of Beads Puzzle

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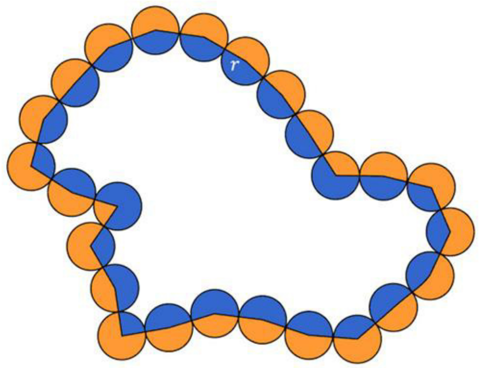

This is a nifty problem[1] from Presh Talwalkar.

This is from a Manga called Q.E.D. I thank Sparky from the Philippines for the suggestion!

A string of beads is formed from 25 circles of the same size. The string passes through the center of each circle. The area enclosed by the string inside each circle is shaded in blue, and the remaining areas of the circles are shaded in orange. What is the value of the orange area minus the blue area? Calculate the area in terms of *r*, the radius of each circle.

My Solution

This problem seems impossible at first. So following Polya's principle from his *How To Solve It* book, I considered a simpler problem. What if the lines of the radii of the circles traced a regular polygon? Figure 1 shows a pattern where the radii trace out an equilateral triangle. We see that the non vertex circles have equal orange and blue areas, so they don't contribute anything to the difference in the areas. Figure 2 - Figure 4 illustrate minimal regular polygonal paths with 3, 4, and 5 sides. The angles α for the blue areas are 60°, 90°, and 108° respectively, or $\pi/3$, $\pi/2$, and $3\pi/5$ respectively in radians.

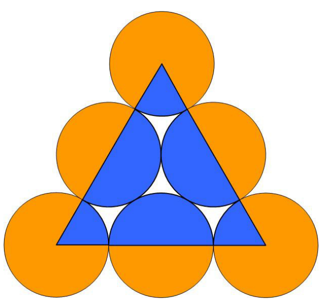

Figure 1

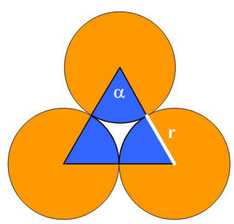

Figure 2

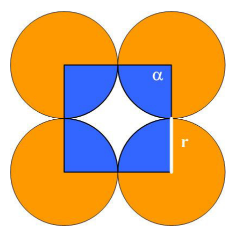

Figure 3

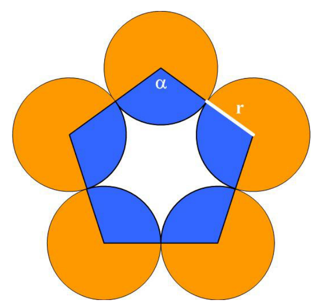

Figure 4

Now consider the areas. Let B be the orange area of one circle and A the area of a blue sector. Then B is the difference in the areas of the entire circle and A, that is, $B = \pi r^2 - A = \pi r^2 - ½\, r^2\alpha$. Now the difference in the orange and blue areas is

$$B - A = \pi r^2 - 2A = \pi r^2 - r^2\alpha$$

To slightly simplify matters, consider α/π. Then

$$B - A = \pi r^2(1 - \alpha/\pi). \qquad (1)$$

Now consider the cases of the first three regular polygonal paths. We get the following total areas:

$\alpha = \pi/3$: $\quad 3(B - A) = 3\pi r^2(1 - (\pi/3)/\pi) = 2\pi r^2$

[1] https://mindyourdecisions.com/blog/2023/09/08/string-of-beads-puzzle/

$\alpha = \pi/2$: $4(B - A) = 4\pi r^2(1 - (\pi/2)/\pi) = 2\pi r^2$

$\alpha = 3\pi/5$: $5(B - A) = 5\pi r^2(1 - (3\pi/5)/\pi) = 2\pi r^2$

So it looks like we get the same answer of $2\pi r^2$ no matter how many sides, but we need to prove that. Let's consider a regular polygonal path of n sides (involving n circles). Now we need a general way to compute the angle α.

Recall a general procedure worked out in other problems,[2] as illustrated in Figure 5. We consider the rotations of a vector about the exterior of the polygon as we trace its path counterclockwise from a starting point back to the starting point. Since the vector makes one complete rotation to return to the starting point, that must be 360° or 2π radians. Thus for an n-sided regular polygon we have

$$n(\pi - \alpha) = 2\pi \quad \text{or} \quad n(1 - \alpha/\pi) = 2 \tag{2}$$

Therefore substituting this relation into equation (1) gives

$$n(B - A) = n\pi r^2(1 - \alpha/\pi) = 2\pi r^2$$

So for any *regular* polygonal path we get the total difference between the orange and blue areas is $2\pi r^2$.

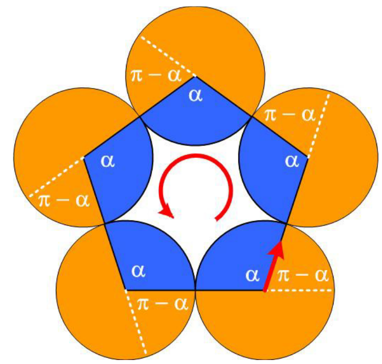

Figure 5

What about irregular polygonal paths, as in our original problem? Then each α angle in the blue sectors would be different, say α_k. Then to get the total difference in orange and blue areas we would have to sum up all the differences of the form in equation (1), that is,

$$\text{Total Orange-Blue area difference} = \sum_{k=1}^{n}(B_k - A_k) = \pi r^2 \sum_{k=1}^{n}(1 - \alpha_k/\pi) \tag{3}$$

and the sum of all the interior angles of the polygonal path would still equal 2π, so that the upgraded equation (2) would become.

$$\sum_{k=1}^{n}(\pi - \alpha_k) = 2\pi \quad \text{or} \quad \sum_{k=1}^{n}(1 - \alpha_k/\pi) = 2 \tag{4}$$

So substituting equation (4) into equation (3) again gives

$$\boxed{\text{Total Orange-Blue area difference} = \sum_{k=1}^{n}(B_k - A_k) = 2\pi r^2}$$

Rather amazing!

Talwalkar Solution

His solution is essentially the same as mine, though he computes it directly without considering regular polygonal paths and he has an alternative justification for computing the α angles.[3]

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[2] http://josmfs.net/2019/02/12/star-sum-of-angles/, https://josmfs.net/2019/03/13/more-sum-of-angles/

[3] JOS: Unfortunately, his method of computing the α_κ angles requires the interior of the string of beads (the interior of the polygonal path of radii) to be *star-connected*, that is there is at least one point inside the polygon from which every point of the polygon can be joined by a straight line lying in the polygon. The turning vector approach I used does not require this, as can be seen by the other problems it was used for.

I should amend my remarks to say as indicated in his figures, he seems to be *assuming* star-connectedness. But if he does not mind overlapping triangles and maintains an orientation that supports negative angles, his method might work, though it is not clear to me without proof.

I should mention there is one other hidden assumption to my solution, and I believe in the problem as well, and that is that the blue areas are always on one side of the traveling tangent vector. That is, the string

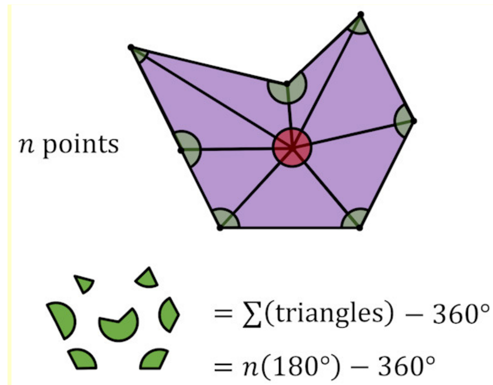
Normally in puzzles like this you will find that both areas are the same so their difference will be 0. That is not the case in this problem! And there's an unexpected result: the difference in areas does not depend on the sum of beads either!

Mathematically we have circles centered at the vertices of a polygon. Actually it may not even be vertices as some centers are just along the edges. So let's figure this out.

The area of a circular sector with central angle θ degrees is:

$$\pi r^2(\theta°/360°)$$

Imagine we can connect n points to form a concave polygon. What is the sum of the angles formed by the polygon edges at the points? Triangulate the polygon into n triangles. Then we have the following diagram:



The sum of the angles at the points is the sum of triangle angles minus the sum of the angles from the central point. The central angles all sum to 360°. Each triangle's angles sum to 180°

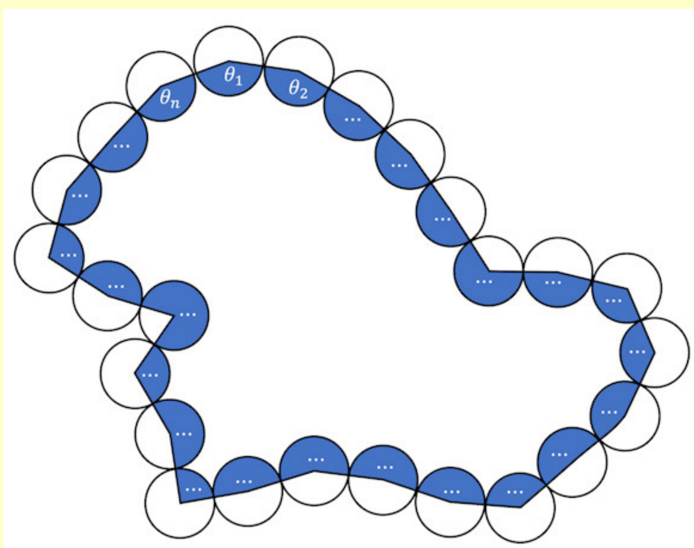
$$\text{sum } n \text{ angles} = \text{sum(triangle angles)} - 360°$$
$$= n(180°) - 360°$$

So now let's solve the problem. To start we have:

blue area + orange area = total area

The total area is the sum of areas of n circles, which is $n\pi r^2$. (The problem has $n = 25$, but we will use a general value n for the moment).

Next let's calculate the blue area. Let the central angles of the circles in degrees be $\theta_1, \theta_2, \ldots, \theta_n$.



Then the sum of the circular sectors is:

$$\pi r^2(\theta_1/360°) + \pi r^2(\theta_2/360°) + \ldots + \pi r^2(\theta_n/360°)$$
$$= (\pi r^2/360°)(\theta_1 + \theta_2 + \ldots + \theta_n)$$

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of beads does not have any twists so that the string crosses itself.

$= (\pi r^2/360°)$(sum of angles of n points)

$= (\pi r^2/360°)(n(180°) - 360°)$

$= \pi r^2(n/2 - 1)$

orange area

= total area – blue area

$= n\pi r^2 - (\pi r^2/360°)(n/2 - 1)$

$= \pi r^2(n/2 + 1)$

Thus we have:

orange area – blue area

$= (\pi r^2/360°)(n/2 + 1) - (\pi r^2/360°)(n/2 - 1)$

$= 2\pi r^2$

Incredibly the difference in areas is exactly the area of 2 circles. This does not depend on the number n at all! We will need at least 3 beads to form a string. But this difference of areas is this same exact value for 3 beads, 4 beads, and even 1 million beads!

The above derivation works even if the circles are not tangent to either other as well–the string just has to pass through the center of circles of equal size.

What an interesting question!