Storm Chaser Problem

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This is a somewhat challenging problem from the 1997 American Invitational Mathematics Exam (AIME) ([1]).

A car travels due east at 2/3 miles per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\sqrt{2}/2$ miles per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $(t_1 + t_2)/2$.

My Solution

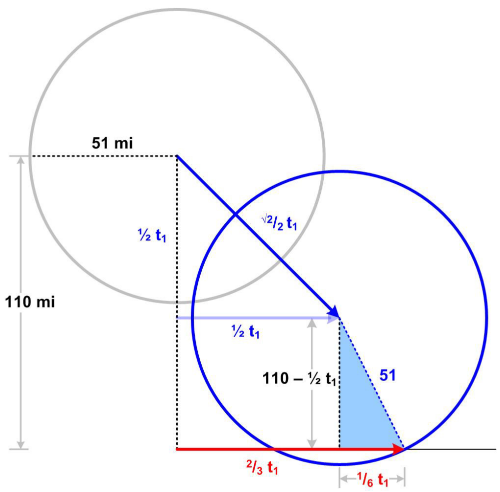

Figure 1

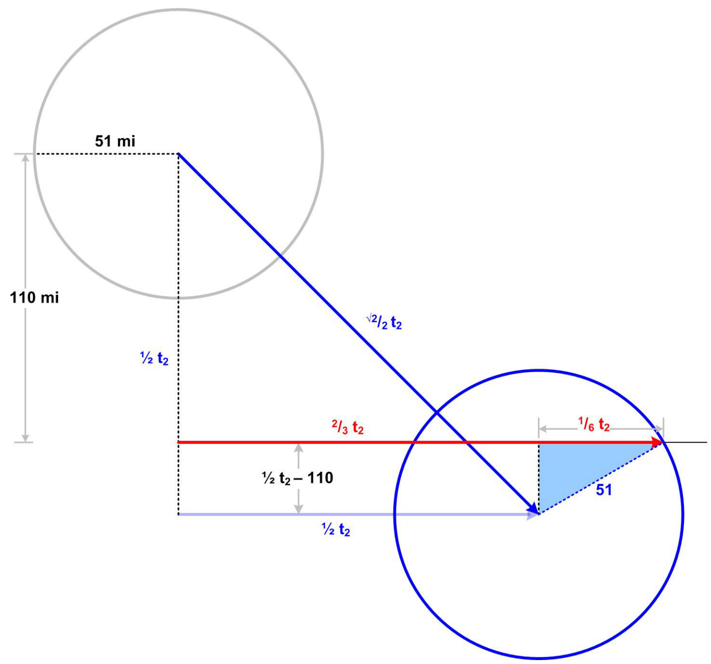

Figure 2

Figure 1 shows the situation when the car first encounters the storm at time t_1. Since the car is moving faster than the eastward speed of the storm, the storm has to catch the car. The difference in the speed of the car and the eastward speed of the storm is $^2/_3 - ½ = {}^1/_6$ miles/min. So the car will be ahead of the center of the storm by $^1/_6\, t_1$ miles. The blue right triangle provides our first equation via the Pythagorean Theorem:

$$51^2 = (110 - ½\, t_1)^2 + (^1/_6\, t_1)^2 = 110^2 - 110\, t_1 + (^1/_4 + {}^1/_{36})t_1^2 \qquad (1)$$

Figure 2 shows the situation when the car leaves the storm circle at time t_2. We get the second equation as before:

$$51^2 = (½\, t_2 - 110)^2 + (^1/_6\, t_2)^2 = 110^2 - 110\, t_2 + (^1/_4 + {}^1/_{36})t_2^2 \qquad (2)$$

Subtracting equation (1) from equation (2) yields

$$5/18\ (t_2^2 - t_1^2) - 110(t_2 - t_1) = 0$$

or, since $t_2 \neq t_1$,

$$5/18\ (t_2 + t_1) = 110$$

or

$$(t_2 + t_1)/2 = 22 \cdot 9 = 198.$$

AIME Solutions

Solution 1

We set up a coordinate system, with the starting point of the car at the origin. At time t, the car is at $({}^2/_3\, t, 0)$ and the center of the storm is at $({}^1/_2\, t, 110 - {}^1/_2\, t)$. Using the distance formula,

$$\sqrt{[({}^2/_3\, t - ½\, t)^2 + (110 - ½\, t)^2]} \leq 51$$

$$({}^1/_4 + {}^1/_{36})t^2 - 110\, t + 110^2 \leq 51^2$$

$${}^5/_{18}\, t^2 - 110\, t + 110^2 - 51^2 \leq 0$$

Noting that $(t_2 + t_1)/2$ is at the maximum point of the parabola [where t_1 and t_2 are the solutions to the quadratic equation (1)], we can use

$$-b/2a = 110 / (2 \cdot {}^5/_{18}) = 198$$ [1]

Solution 2 (more formal explanations for Solution 1)

First do the same process for assigning coordinates to the car. The car moves ${}^2/_3$ miles per minute to the right, so the position starting from (0, 0) is $({}^2/_3\, t, 0)$.

Take the storm as circle. Given southeast movement, split the vector into component, getting position $({}^1/_2\, t, 110 - {}^1/_2\, t)$ for the storm's center. This circle with radius 51 yields

$$(x - {}^1/_2\, t)^2 + (y - 110 + {}^1/_2\, t)^2 = 51^2$$

Now substitute the car's coordinates into the circle's:

$$({}^2/_3\, t - {}^1/_2\, t)^2 + (-110 + {}^1/_2\, t)^2 = 51^2$$

Simplifying and then squaring:

$$({}^1/_6\, t)^2 + (-110 + {}^1/_2\, t)^2 = 51^2$$

$$({}^1/_4 + {}^1/_{36})t^2 - 110\, t + 110^2 = 51^2$$

Forming into a quadratic we get the following, then set equal to 0, since the first time the car hits the circumference of the storm is t_1 and the second is t_2.

$${}^5/_{18}\, t^2 - 110\, t + 110^2 - 51^2 = 0$$

The problem asks for sum of solutions divided by 2 so sum is equal to:

$$-b/2a = 110 / (2 \cdot {}^5/_{18}) = 198$$ [2]

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[1] JOS: If we convert the general quadratic $ax^2 + bx + c = 0$ into the monic polynomial equation $x^2 + (b/a)x + c/a = 0$, then the coefficient of the linear term is the negative sum of the roots t_1 and t_2. That is, $(t_1 + t_2) = -b/a$.

[2] JOS: Same argument as in Note 1.

Solution 3 (No Coordinates)

We only need to know how the storm and car move relative to each other, so we can find this by subtracting the storm's movement vector from the car's. This gives the car's movement vector as $(^1/_6, ^1/_2)$. Labeling the car's starting position A, the storm center B, and the right triangle formed by AB with a right angle at B and the car's path, we get the following diagram [Figure 3], with AD as our desired length since D is the average of the points where the car enters and exits the storm.

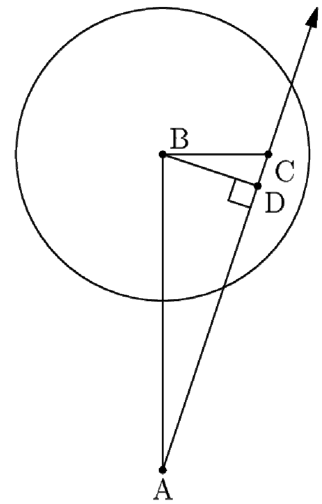

Figure 3

AB = 110, so CB = 110/3. The Pythagorean Theorem then gives

$$AC = 110\sqrt{10} / 3,$$

and since $\Delta ABC \sim \Delta ADB$,

$$AD = AB \cdot AB/AC = 33\sqrt{10}.$$

The Pythagorean Theorem now gives the cars speed as $\sqrt{(5/18)}$ and finally

$$AD = 33\sqrt{10} / \sqrt{(5/18)} = 198.$$

References

[1] "Problem 7" 1997 AIME Problems (https://artofproblemsolving.com/wiki/index.php/1997_AIME_Problems)