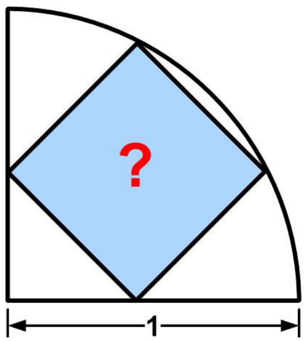
Square In A Quarter Circle

30 May 2022

Jim Stevenson



Another puzzle by Presh Talwalkar ([1]).

Thanks to John H. for the suggestion!

A square is inscribed in a quarter circle such that the outer vertices are on the arc of the quarter circle. If the quarter circle has a radius equal to 1, what is the area of the square?

I am told this was given to 7th grade students (ages 12-13), and I think it is a very challenging problem for that age group. In fact I think it is a good problem for any geometry student.

My Solution

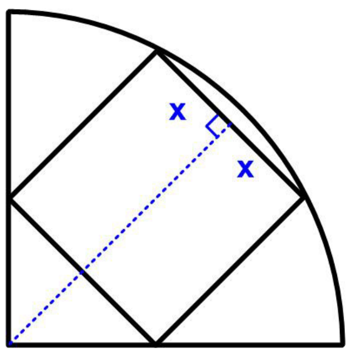


Figure 1

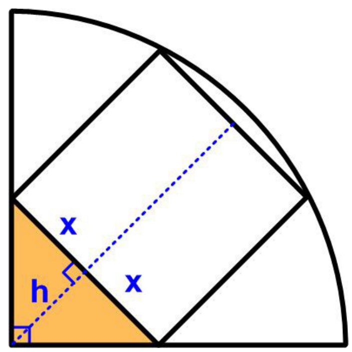


Figure 2

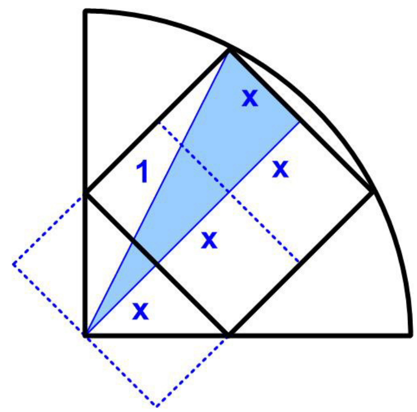


Figure 3

First we take the perpendicular bisector of the edge of the square intersecting the quarter circle arc. Since the edge is a chord of the circle, the perpendicular bisector passes through the center of the circle (Figure 1). Since the inscribed figure is a square, this perpendicular bisector is parallel to the sides of the square and so is a perpendicular bisector of the other edge as well. Therefore the segment of the bisector in the (orange) right triangle is an altitude h of the triangle (Figure 2). From the geometric mean we get

$$x/h = h/x \quad \Rightarrow \quad h^2 = x^2 \quad \Rightarrow \quad h = x.$$

Therefore, the perpendicular bisector is of length $3x$, which gives us a (blue) right triangle with sides x, $3x$, and hypotenuse 1 (Figure 3). So by the Pythagorean Theorem we have

$$9x^2 + x^2 = 1 \quad \Rightarrow \quad x^2 = {}^1/_{10}$$

or

$$\text{Area of square} = 4\,x^2 = {}^2/_5.$$

Talwalkar Solution

Talwalkar's solution is similar to mine, but he used a different argument to obtain the length of the perpendicular bisector inside the orange right triangle.

Side BC is a chord of circle O [Figure 4]. Construct the perpendicular bisector of BC. It will pass through the center O (since O is equidistant from B and C), and label its intersection with BC as E and its intersection with AD as F. Because $ABCD$ is a square, the perpendicular bisector of BC is also the perpendicular bisector of AD. Triangles AFO and DFO are congruent by side-angle-side since $AF = FD$, angles AFO and DFO are right angles, and $OF = OF$. Thus $AOF = DOF$, and since $AOD = 90°$, we have $AOF = DOF = 45°$. Thus OFD is an isosceles right triangle with $OF = FD$.

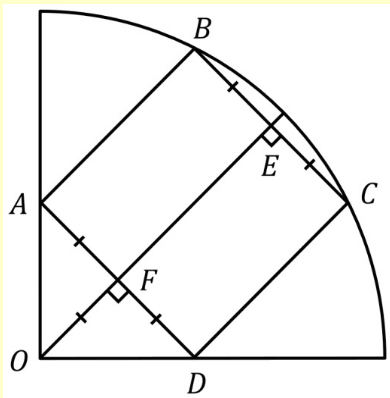


Figure 4

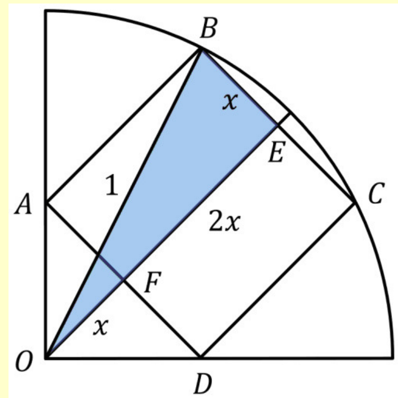


Figure 5

Construct the radius OB which has length 1. Suppose half of the square's side has length equal to x, so that $BE = FD = FO = x$ and $FE = CD = 2x$ [Figure 5]. Then OEB is a right triangle with legs x, $x + 2x = 3x$, and a hypotenuse equal to 1. Thus we have:

$$x^2 + (3x)^2 = 1^2$$

$$10x^2 = 1$$

$$x^2 = 1/10$$

The square's area is then:

$$(2x)^2 = 4x^2 = 4(1/10) = 4/10 = 0.4$$

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References

I thank these people who helped me prove the first step on Twitter:

@ahmed_elashraf https://twitter.com/ahmed_elashraf/status/1450213493798494211

@ScottRollison https://twitter.com/ScottRollison/status/1450251161798430728

@OkanAtalay1970 https://twitter.com/OkanAtalay1970/status/1450305206378434562

References

[1] Talwalkar, Presh, "Square in a Quarter Circle," *Mind Your Decisions*, 7 December 2021 (https://mindyourdecisions.com/blog/2021/12/07/square-in-a-quarter-circle/)