Pythagorean Parabola Puzzle

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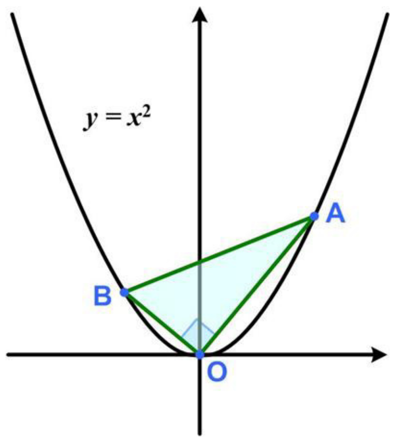

Since the changes in Twitter (now X), I have not been able to see the posts, not being a subscriber. But I noticed poking around that some twitter accounts were still viewable. However, like some demented aging octogenarian they had lost track of time, that is, instead of being sorted with the most recent post first, they showed a random scattering of posts from different times. So a current post could be right next to one several years ago. That is what I discovered with the now defunct MathsMonday site. I found a post[1] from 10 May 2021 that I had not seen before, namely,

The points A and B are on the curve $y = x^2$ such that AOB is a right angle. What points A and B will give the smallest possible area for the triangle AOB?

Solution

I first tried to solve the problem geometrically. *Suppose*, as the point A moved up the right-hand branch of the parabola, the areas grew. Flipping the triangle about the y-axis produced a symmetric triangle with the same area. Therefore, moving A back down the right-hand branch until it was opposite the point B would shrink the areas and arrive at a minimum. This would mean the side OA was along a 45° line and so would intersect the parabola at the point A = (1, 1). Therefore B = (–1, 1). *But*, I could not see how to prove easily that the areas did grow as A moved up the branch. Also I hadn't explicitly used the fact that the lines were perpendicular. So I reverted to analytic geometry and calculus.

From Figure 1 we see that the area of the triangle $\mathcal{A}$ is $d_1d_2/2$. Let $\mathcal{B} = (2\mathcal{A})^2$. Then $\mathcal{B}$ is a minimum where $\mathcal{A}$ is. We have

$$\mathcal{B} = (d_1d_2)^2 = (x_1^{\,2} + y_1^{\,2})\,(x_2^{\,2} + y_2^{\,2}). \tag{1}$$

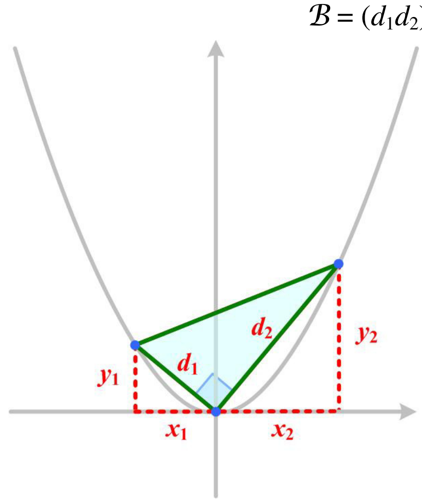

Figure 1

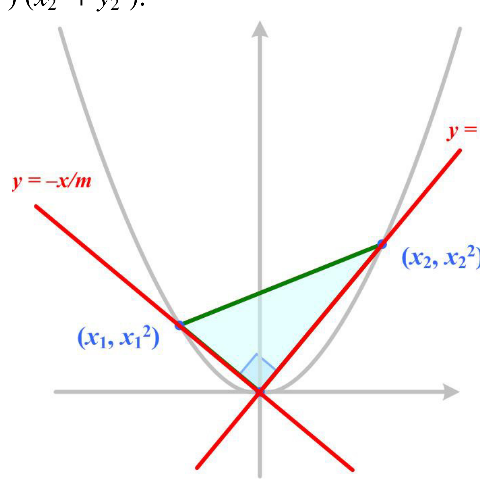

Figure 2

[1] https://twitter.com/MEIMaths/status/1391679408310784003

Now add the perpendicular straight lines shown in Figure 2. They intersect the parabola at points A and B and have the equations $y = mx$ and $y = -x/m$ for some slope $m > 0$. Therefore we have

$$y_2 = mx_2 = x_2^2 \Rightarrow x_2 = m \quad \text{and} \quad y_1 = -x_1/m = x_1^2 \Rightarrow x_1 = -1/m$$

Plugging these values into equation (1) yields

$$\mathcal{B}(m) = (1/m^2 + 1/m^4)\,(m^2 + m^4) = 2 + m^2 + 1/m^2.$$

Now take the derivative of $\mathcal{B}(m)$ with respect to m and set the result to 0:

$$\mathcal{B}'(m) = 2m - 2/m^3 = 0$$

Therefore, $m = 1$ (assuming $m > 0$), that is, the line lies along the 45° line as we conjectured. To prove this is a minimum, take the second derivative:

$$\mathcal{B}''(m) = 2 + 6/m^4 > 0$$

So in particular $\mathcal{B}''(1) = 8 > 0$, which implies $\mathcal{B}(m)$ is concave up at $m = 1$, and so a minimum.

Therefore, as before, $m = 1$ implies A = (1, 1) and B = (–1, 1).