Peirce's Law

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Wikipedia

The June 2023 *Carnival of Mathematics* # 216[1] at Eddie's Math and Calculator Blog has the rather arresting item concerning Peirce's Law from the American logician Charles Sanders Peirce[2] (1839 – 1914).

Peirce's Law: Jon Awbrey of the Inquiry Into Inquiry blog ([1])

This article explains Pierce's Law and provides the proof of the law. The proof is provided in two ways: by reason and graphically. Simply put, for propositions P and Q, the law states:

P must be true if there exists Q such that the statement "if P then Q" is true. In symbols:

$$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$$

The law is an interesting tongue twister to say the least.

Perhaps another way of saying it is "if the implication $P \Rightarrow Q$ implies that P is true, then P must be true." Still, it sounds weird.

Proof

Rather than follow Aubrey's presentation of the original, I think it is easier to use truth tables.[3] Recall that $P \Rightarrow Q$ is logically equivalent to $\sim(P \wedge \sim Q)$, that is, "if P is true, then Q is true" is equivalent to "what must never happen is for P to be true and Q false," that is, $\sim(P \wedge \sim Q)$. Here is the truth table to show $(P \Rightarrow Q) \Leftrightarrow \sim(P \wedge \sim Q)$, where $A \Leftrightarrow B$ means A is true whenever B is true and false whenever B is false, or A is true if and only if B is true.

(P	⇒	Q)	⇔	~ (	P	∧	(~	Q))
(1)	(2)	(1)	(5)	(4)	(1)	(3)	(2)	(1)
T	T	T	T	T	T	F	F	T
F	T	T	T	T	F	F	F	T
T	F	F	T	F	T	T	T	F
F	T	F	T	T	F	F	T	F

Recall that the numbers in parentheses at the top of the columns represent the sequential steps in filling in the truth values, where step (1) is assigning all T, F combinations to statements P and Q. Since the two statements each can have two values, there are four possible combinations and so four rows in the table. Step (2) gives the T, F values for $P \Rightarrow Q$ and ~Q respectively. Step (3) resolves

1 https://edspi31415.blogspot.com/2023/06/carnival-of-mathematics-216.html, retrieved 6/13/2023

2 JOS: Pronounced "purse". Note the "e" comes before the "i".

3 JOS: For more explanation see the "Appendix: Vacuous Truth" in my post "Pinocchio's Hats" (https://josmfs.net/2022/07/09/pinocchios-hats/).

P ∧ ~Q, and step (4) ~(P ∧ ~Q). Finally step (5) shows the equivalence holds for all values of P and Q, that is, we have a tautology.

Then the truth table for ((P ⇒ Q) ⇒ P) ⇒ P is

((P	**⇒**	**Q)**	**⇒**	**P)**	**⇒**	**P**
(1)	**(2)**	(1)	**(3)**	(1)	**(4)**	(1)
T	**T**	T	**T**	T	**T**	T
F	**T**	T	**F**	F	**T**	F
T	**F**	F	**T**	T	**T**	T
F	**T**	F	**F**	F	**T**	F

Again, the fact that the final result in the truth table is a tautology (always true) means that Peirce's Law holds no matter what the values of P and Q are.

References

[1] Awbrey, Jon, "A Curious Truth of Classical Logic", *Inquiry into Inquiry*, 6 October 2008. (https://inquiryintoinquiry.com/2008/10/06/peirces-law/)