A Triangle Puzzle

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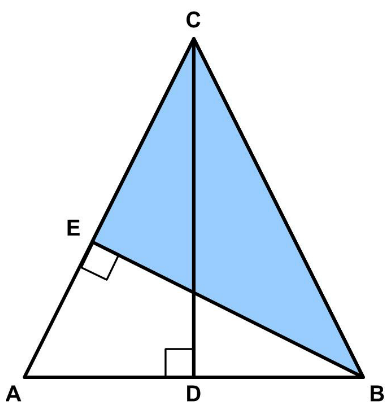

The Futility Closet website had the following problem:[1]

In isosceles triangle ABC, CD = AB and BE is perpendicular to AC. Show that CEB is a 3-4-5 right triangle.

Futility Closet Solution

Place triangle ABC into a square and draw the grid shown here, in which each line is parallel to EB or AC and contains a midpoint or quarter point on the side of the square. With CE and BE having lengths 3 and 4, we can determine that hypotenuse CB is of length 5.

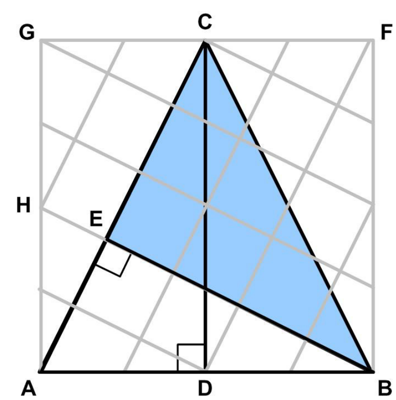

Figure 1

"We have therefore shown the nature of this right triangle using a cleverly constructed diagram, and we have avoided lots of tedious computation," write Alfred S. Posamentier and Ingmar Lehmann in *Mathematical Curiosities*. "This is an example where we can see how an elegant solution underscores the beauty of mathematics."

My Solution

The Futility Closet solution is intriguing, but I feel it may harbor defects that lead similar geometric constructions to geometric paradoxes. For example, how do you know that the line EB extended to the edge of the square at H actually bisects the edge of the square? It "looks" like it does, but that is not sufficient for a real proof.

(Actually, the construction can be made rigorous. Notice that EB extended, EH, is the hypotenuse of the right triangle ABH, which happens to be congruent to the right triangle ADC via a clockwise 90° rotation. Since AD will be shown to be half the edge of the square AB, then AH is also half the edge, which means H *is* the midpoint of AG. Marking off the other parallel lines through midpoints of the half edges, plus their rotated versions, produces the grid as shown. Arguing about how lines parallel to the base of a triangle cut off proportional lengths on the sides of the triangles yields the evenly spaced segments (and thus squares) of the slanted lines. But none of this discussion was given in the Futility Closet solution.)

I have created an alternative numerical demonstration that proves the 3-4-5 claim in a simple, direct way.

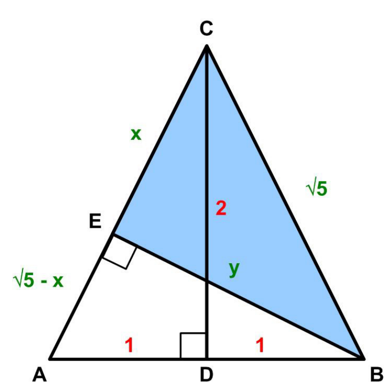

Figure 2

Consider Figure 2. Since the triangle ABC is isosceles where CB = AC, then CD is a perpendicular bisector of AB. Thus if we assign arbitrarily the value of 2 to the length of CD and since CD =

[1] https://www.futilitycloset.com/2018/07/21/a-triangle-puzzle/

AB, then AD = DB = 1, as shown. The Pythagorean Theorem yields $\sqrt{5}$ as the length of CB.

Now assign the values of x and y to the legs of the blue right triangle CEB. Then again by the Pythagorean Theorem we have

$$x^2 + y^2 = 5$$

and

$$(\sqrt{5} - x)^2 + y^2 = 4$$

Subtracting the second equation from the first yields

$$1 = x^2 - (\sqrt{5} - x)^2 = x^2 - (5 - 2\sqrt{5}x + x^2) = 2\sqrt{5}x - 5$$

or

$$x = 3 / \sqrt{5}.$$

Then from the first equation,

$$y^2 = 5 - x^2 = 5 - (3 / \sqrt{5})^2 = 16/5$$

so that

$$y = 4 / \sqrt{5}$$

Since $\sqrt{5} = 5 / \sqrt{5}$, we have the blue triangle CEB is a 3-4-5 triangle.