Putnam Ellipse Areas Problem

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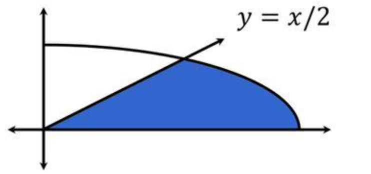

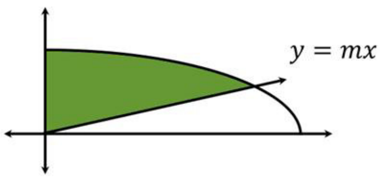

This is a nifty problem from Presh Talwakar.[1]

This is adapted from the 1994 Putnam, A2. Thanks to Nirman for the suggestion!

Let R be the region in the first quadrant bounded by the x-axis, the line $y = x/2$, and the ellipse $x^2/9 + y^2 = 1$. Let R' be the region in the first quadrant bounded by the y-axis, the line $y = mx$ and the ellipse. Find the value of m such that R and R' have the same area.

Solution

Applying one of Polya's ideas from *How to Solve It*, we consider a simpler problem, namely, what if the ellipses were circles? Then the equal areas would be symmetric about the 45° line. Using an idea employed in the solution of the "Octagonal Area Problem",[2] we compress the horizontal scale to create a regular figure—in this case, a circle. This compresses all areas by the scale reduction. But the areas are restored to their original values when the reverse scale expansion factor is applied to restore the original figures. Note that equal areas are preserved under both the linear compression and linear expansion. The details are as follows (Figure 1).

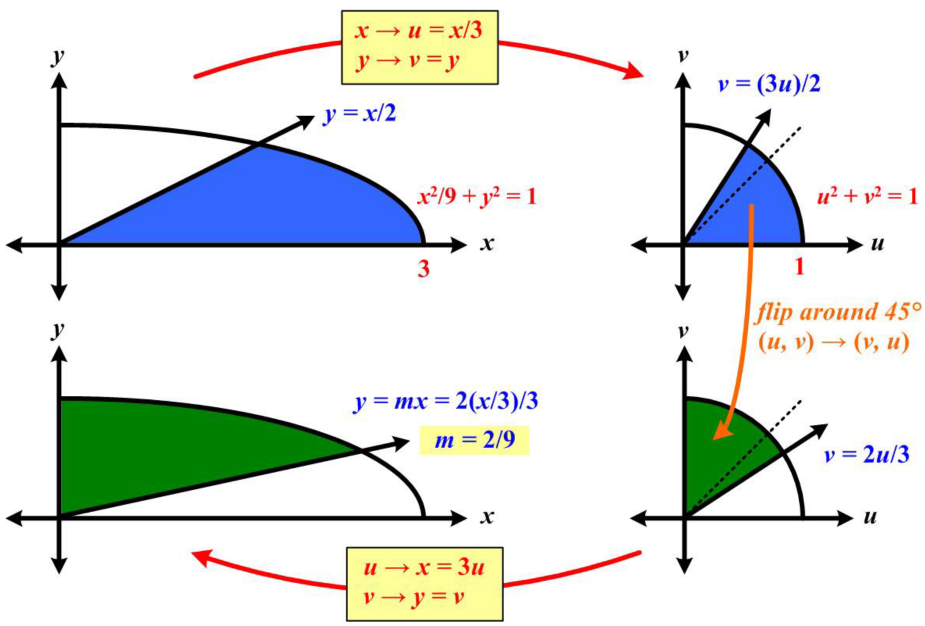

Figure 1

We make the coordinate transformations from the (x, y) coordinates to the (u, v) coordinates via $u = x/3$ and $v = y$. Then in the new coordinates the ellipse $x^2/9 + y^2 = 1$ becomes the circle $u^2 + v^2 = 1$, and the straight line $y = x/2$ becomes the line $v = (3u)/2$. Flipping the blue area about the 45° line is

[1] https://mindyourdecisions.com/blog/2023/04/24/putnam-ellipse-areas-question/
[2] https://josmfs.net/2022/01/29/octagonal-area-problem/

accomplished by swapping the coordinates $(u, v) \rightarrow (v, u)$. So now the line $v = 3u/2$ becomes the line $u = 3v/2$ or $v = 2u/3$. Transforming back to the xy-coordinates yields

$$y = v = 2u/3 = 2(x/3)/3 = 2x/9.$$

Therefore, $m = 2/9$, which is the desired slope providing the equal areas.

(Talwalkar's solution[3] turns out to be virtually the same, and so will be omitted.)

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[3] Putnam 1994, A2 solution, John Scholes (https://prase.cz/kalva/putnam/psoln/psol942.html)