The Tired Messenger Problem

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Here is another challenging problem from the Polish Mathematical Olympiads ([1]). Its generality will cause more thought than for a simpler, specific problem.

A cyclist sets off from point O and rides with constant velocity v along a rectilinear highway. A messenger, who is at a distance a from point O and at a distance b from the highway, wants to deliver a letter to the cyclist. What is the minimum velocity with which the messenger should run in order to attain his objective?

My Solution

Case 1. Figure 1 shows the general setting for the problem for the case where the cyclist heads toward the messenger. The point on the horizontal road where the messenger running at speed v_M meets the cyclist traveling at speed v_C is labeled x. Since the distances a and b are not specified, we need to consider all values and how they affect the problem.

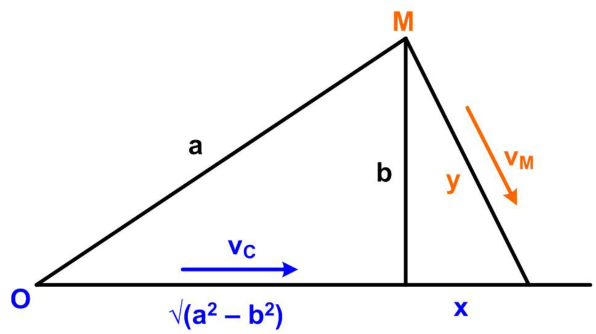

Figure 1 Case 1

First, note that $0 \le b \le a$, since b is the minimum distance to the road. If $b = 0$, then the messenger is already on the road and can just wait for the cyclist, in which case his minimum speed is $v_M = 0$. So assume $b > 0$. Furthermore, if $b = a$, the cyclist is no longer heading toward the messenger, so we will postpone that situation to Case 2. Therefore we consider $0 < b < a$.

The times taken for the cyclist and the messenger to meet are the same, say time t. Consider the distances traveled by each in that time:

$$y = \sqrt{b^2 + x^2} = v_M t \text{ and } \sqrt{a^2 - b^2} + x = v_C t$$

Eliminating t yields

$$v_M = \frac{\sqrt{b^2 + x^2}}{\sqrt{a^2 - b^2} + x} v_C = r(x) v_C$$

So v_M is minimal when $r(x)$ is. Differentiate $r(x)$ with respect to x and set it to zero. Then

$$r'(x) = \frac{x\sqrt{a^2 - b^2} - b^2}{\sqrt{b^2 + x^2}\left(\sqrt{a^2 - b^2} + x\right)^2} = 0$$

when
$$x = \frac{b^2}{\sqrt{a^2 - b^2}} \quad .$$

Now $r'(x) < 0$ when $x < b^2/\sqrt{a^2-b^2}$ and $r'(x) > 0$ when $x > b^2/\sqrt{a^2-b^2}$. Therefore $x = b^2/\sqrt{a^2-b^2}$ is a minimum point for $r(x)$, and at this point the minimum value of $r(x)$ is b/a. So the minimum speed for the messenger v_M is given by

$$\boxed{v_M = \frac{b}{a} v_C}$$

After checking the Olympiad solution I realized my parameterization obscured a significant result, namely that the messenger's path to the road should be perpendicular to line OM. Figure 2 shows the values of the lengths of the paths when the minimal solution for x ($x = b^2/\sqrt{a^2-b^2}$) is substituted. Then

$$a^2 + \left(\frac{ab}{\sqrt{a^2-b^2}}\right)^2 = \left(\frac{a^2}{\sqrt{a^2-b^2}}\right)^2 = \left(\sqrt{a^2-b^2} + \frac{b^2}{\sqrt{a^2-b^2}}\right)^2$$

means the sum of the squares of the legs equals the square of the hypotenuse, and so the triangle is a right triangle.

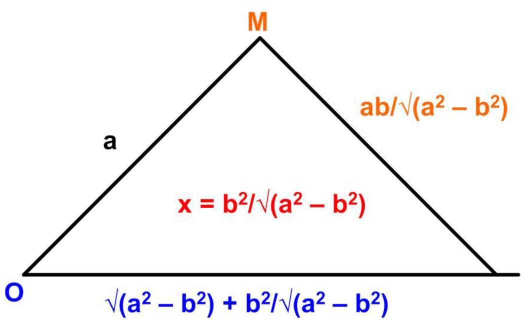

Figure 2

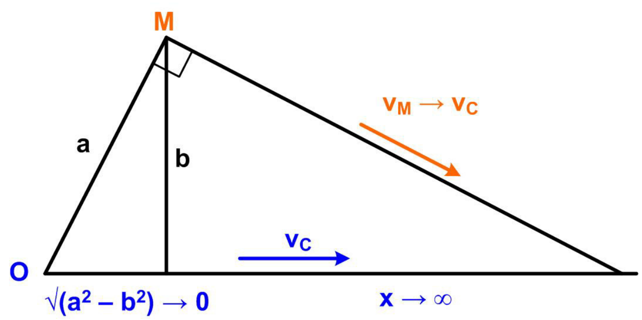

Figure 3

As a approaches b, the point of meeting at x with minimal speed for the messenger of $v_M = (b/a)v_C$ moves further and further away, approaching infinity (Figure 3), and the slower speed of the messenger would approach the speed of the cyclist.

Case 2. Figure 4 shows the general setting for the problem for the case where the cyclist heads away from the messenger. Now we assume $0 < b \le a$. that is, we include the case when $a = b$. We then have

$$y = \sqrt{b^2 + (\sqrt{a^2-b^2} + x)^2} = v_M t$$

$$x = v_C t$$

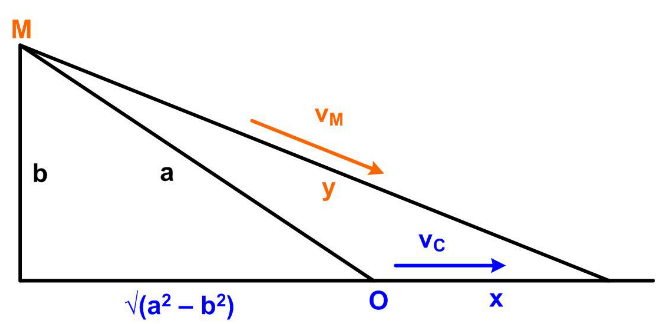

Figure 4 Case 2

Eliminating t yields

$$v_M = \frac{\sqrt{b^2 + (\sqrt{a^2-b^2}+x)^2}}{x} v_C = \frac{\sqrt{a^2 + 2x\sqrt{a^2-b^2} + x^2}}{x} v_C = r(x) v_C$$

Now

$$r'(x) = -\frac{a^2 + x\sqrt{a^2 - b^2}}{x^2\sqrt{a^2 + 2x\sqrt{a^2 - b^2} + x^2}} < 0$$

for all $x > 0$. Therefore $r(x)$ has no minimum for $x > 0$. Furthermore,

$$r(x) = \sqrt{\frac{a^2}{x^2} + \frac{2\sqrt{a^2 - b^2}}{x} + 1} > 1$$

and $r(x) \to 1$ as $x \to \infty$. (Also $r(x) \to \infty$ as $x \to 0$.) So the messenger will always have to run faster than the cyclist pedals to meet him at any point x, and his speed approaches that of the cyclist as the meeting point moves further away.

Olympiads Solution

Again I provide the images of the Olympiads solutions on p.4 below. The Olympiads solutions avoid calculus.

References

[1] Straszewicz, S., *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads*, J. Smolska, tr., Popular Lectures in Mathematics, Vol.12, Pergamon Press, London, 1965 (Polish edition 1960). Problem 153, solution p.348

Olympiad Solution

153. We shall assume that $b > 0$; let the reader himself formulate the answer to the question asked if $b = 0$, i.e. if the messenger is on the road.

Method I. Let M denote the point at which the messenger finds himself, S the point of the meeting, t the time which will elapse between the initial moment and the moment of the meeting and x the velocity of the messenger. Applying the Cosine Rule to triangle MOS, in which $OS = vt$, $MS = xt$, $OM = a$, we obtain

$$x^2t^2 = a^2+v^2t^2-2avt \cos \alpha,$$

where α denotes angle MOS. Hence

$$x^2 = \frac{a^2}{t^2} - 2av \cos \alpha \times \frac{1}{t} + v^2.$$

Let us write $1/t = s$; then

$$x^2 = a^2s^2 - 2av \cos \alpha \times s + v^2 = (as - v \cos \alpha)^2 + v^2 - v^2 \cos^2 \alpha,$$

or, more briefly,

$$x^2 = (as - v \cos \alpha)^2 + v^2 \sin^2 \alpha. \qquad (1)$$

We seek a *positive* value of s for which the positive quantity x, and thus also x^2, has the least value. We must distinguish two cases here:

Case 1: $\cos \alpha > 0$, i.e. α is an acute angle. It follows from formula (1) that x has the least value $x_{\min}$ if $as - v \cos \alpha = 0$, whence

$$s = \frac{v \cos \alpha}{a}.$$

Then

$$x_{\min}^2 = v^2 \sin^2\alpha, \quad \text{and thus} \quad x_{\min} = v \sin \alpha.$$

Case 2: $\cos \alpha \leqslant 0$, i.e. α is a right angle or an obtuse one. In this case the required minimum does not exist since $as - v \cos \alpha > 0$, and thus also x^2 is the smaller the nearer s is to zero, i.e. the greater is t. As t increases indefinitely, s tends to zero and x, as shown by formula (1), tends to v.

We shall explain these results with the aid of a drawing. If $\alpha < 90°$ (Fig. 252), the minimum velocity of the messenger is equal to $v \sin \alpha = vb/a$; the meeting will take place at the moment when $1/t = (v \cos \alpha)/a$. Then

$$MS = v \sin \alpha \frac{a}{v \cos \alpha} = a \tan \alpha,$$

which means that $\sphericalangle OMS = 90°$; the messenger should run along a perpendicular to OM.

If $\alpha \geqslant 90°$ (Fig. 253), the messenger must cover a longer route than the cyclist, and thus he can overtake him only if his velocity is greater than the velocity v of the cyclist; the necessary surplus of velocity, however, will be the less the greater is the angle

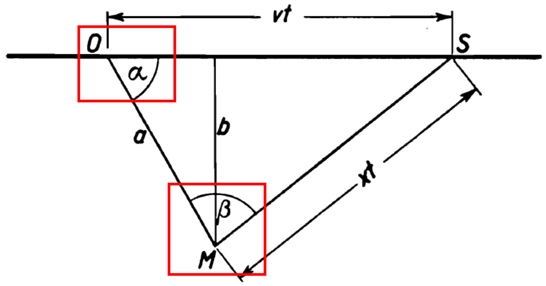

FIG. 252

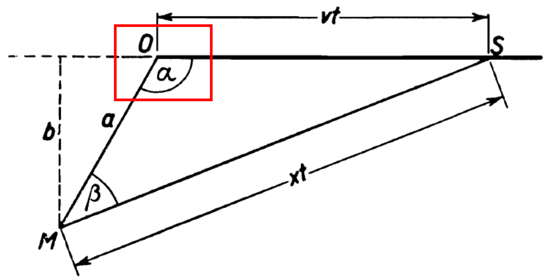

FIG. 253

$\measuredangle OMS = \beta$, and can be arbitrarily small if the cyclist rides in a direction which forms with OM an angle sufficiently near $180° - \alpha$.

Method II. Adopting the same notation as before, we have

$$\frac{x}{v} = \frac{xt}{vt} = \frac{MS}{OS},$$

whence by the Sine Rule (Fig. 252)

$$\frac{x}{v} = \frac{\sin \alpha}{\sin \beta} \quad \text{and} \quad x = \frac{\sin \alpha}{\sin \beta} \times v.$$

This equality implies that x assumes the least value when $\sin \beta$ is greatest. If $\alpha < 90°$, this occurs for $\beta = 90°$, whence

$$x_{\min} = v \sin \alpha = \frac{vb}{a}.$$

$$v_M = \frac{b}{a} v_C$$

If $\alpha \geqslant 90°$ (Fig. 253), then angle β is acute; a greatest value of β does not exist, the velocity x is the smaller the nearer the angle β is to $180° - \alpha$. As angle β increases and tends to $180° - \alpha$, velocity x decreases and tends to v.

REMARK. In the above solution we can dispense with the use of trigonometry, reasoning as follows.

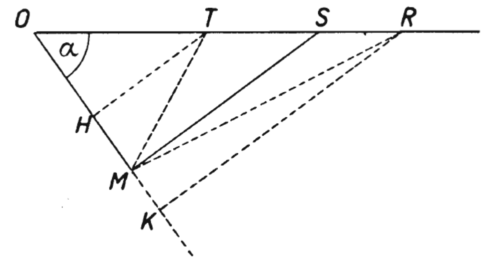

FIG. 254

If $\alpha < 90°$ (Fig. 254) and $MS \perp OM$, we draw $TH \perp OM$ and $RK \perp OM$. Then

$$\frac{MS}{OS} = \frac{HT}{OT} < \frac{MT}{OT}, \qquad \frac{MS}{OS} = \frac{KR}{OR} < \frac{MR}{OR},$$

whence at point S of the road the ratio of the distances from points M and O is smallest.

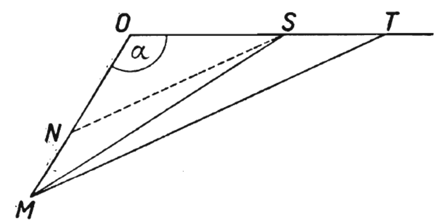

FIG. 255

If $\alpha \geqslant 90°$ and point T lies farther from point O than point S (Fig. 255), then, drawing NS parallel to MT, we have

$$\frac{MT}{OT} = \frac{NS}{OS} < \frac{MS}{OS},$$

whence it is obvious that the required minimum does not exist.

Method III. We are to find on the road a point S at which the ratio MS/OS has its minimum.

Now every point S of the road lies on the circle of Apollonius constructed for the segment OM and ratio $k = MS/OS$. If $\alpha \geqslant 90°$, then k is greater than 1, and is the smaller (the nearer to 1) the greater the Apollonius circle; thus the required minimum does not exist.

If $\alpha < 90°$, the least value of k is less than 1 and corresponds to that circle of Apollonius which is tangent to the road. Let T be the centre of this circle and let K and L be the points at

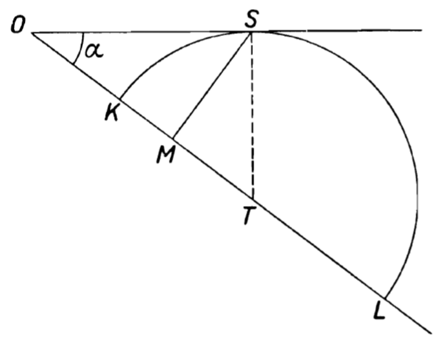

FIG. 256

which the circle intersects the straight line OM (Fig. 256). The pairs of points K, L and O, M separate each other harmonically, whence

$$TK^2 = TO \times TM,$$

and since

$$TK = TS,$$

we have

$$TS^2 = TO \times TM,$$

whence we conclude that point M is the orthogonal projection of point S upon the straight line OM and $MS/OS = \sin\alpha = b/a$.

REMARK. In the above reasoning we have made use of the following theorem:

If the pairs of points A, B and C, D separate each other harmonically, i.e. if $AC:CB = AD:BD$ and P is the mid-point of the segment AB (Fig. 257), *then*

$$PB^2 = PC \times PD.$$

This equality may be proved as follows: by hypothesis we have

$$AC \times BD = CB \times AD.$$

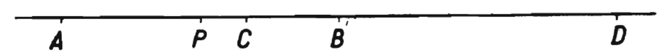

FIG. 257

We replace the segments appearing in this equality by segments with the initial point P, for instance $AC = AP+PC = PB+PC$, $BD = PD-PB$, etc.; we obtain

$$(PB+PC)(PD-PB) = (PB-PC)(PB+PD),$$

which, suitably arranged, gives

$$PB^2 = PC \times PD.$$

The same calculation performed in the inverse order shows that, if the above equality holds, the pairs of points A, B and C, D separate each other harmonically.