Pillar Wrapping Problem

7 April 2023

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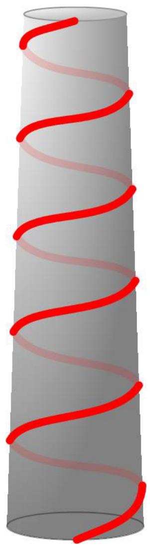

This is a fun problem from the 1949 *Eureka* magazine ([1]).

The following problems were set at the Archimedeans' 1949 Problems Drive. Competitors were allowed five minutes for each question. [This is problem #9.]

A pillar is in the form of a truncated right circular cone. The diameter at the top is 1 ft., at the bottom it is 2 ft. The slant height is 15 ft. A streamer is wound exactly five times round the pillar starting at the top and ending at the bottom. What is the shortest length the streamer can have?

My Solution

In flat Euclidean geometry the shortest distance between two points is along the straight line joining the points. This is called a *geodesic*. On curved surfaces defining a straight line becomes a problem. If we could somehow reduce the curved surface to a flat plane, then it would be easy to find the geodesic.

It just so happens that there are certain surfaces that appear curved but have intrinsic curvature of zero. This intrinsic curvature is also called *Gaussian curvature*, which was discussed in some detail in the posting "Bugles, Trumpets, and Beltrami."[1] It is the product of the two orthogonal principle curvatures. For a cone and a cylinder one of the principle curvatures is zero, since the corresponding principle direction lies along a line in the surface. Surfaces generated by a line moving along a curve are called *ruled surfaces*. Not all ruled surfaces have zero intrinsic curvature. For example, the hyperboloid of one sheet discussed in "Hyperboloid as Ruled Surface"[2] does not have zero intrinsic curvature. Ruled surfaces with intrinsic curvature zero are called *developable surfaces*.

Why these surfaces are important is that we can cut them along their generating line and then unwrap the surface so that it can lie on a flat plane. So we do that with the surface of the pillar in our problem (Figure 1). Then we have

$$r\theta = \pi \text{ and } (r + 15)\theta = 2\pi$$

so that

$$\theta = \pi/15 \text{ and } r = \pi\,(15/\pi) = 15.$$

Now since we are wrapping the streamer around the pillar 5 times, we want 5 images of the flattened pillar, so that $5\theta = \pi/3 = 60°$ (Figure 2).

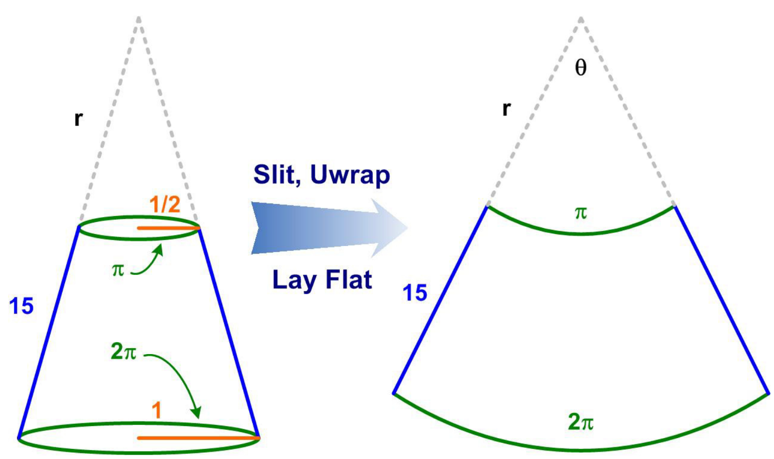

Figure 1

[1] "Bugles, Trumpets, and Beltrami" http://josmfs.net/2019/02/24/bugles-trumpets-and-beltrami/

[2] "Hyperboloid as Ruled Surface" http://josmfs.net/2019/01/30/hyperboloid-as-ruled-surface/

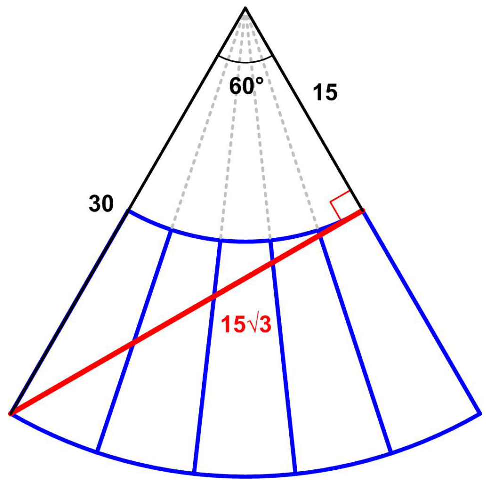

Figure 2

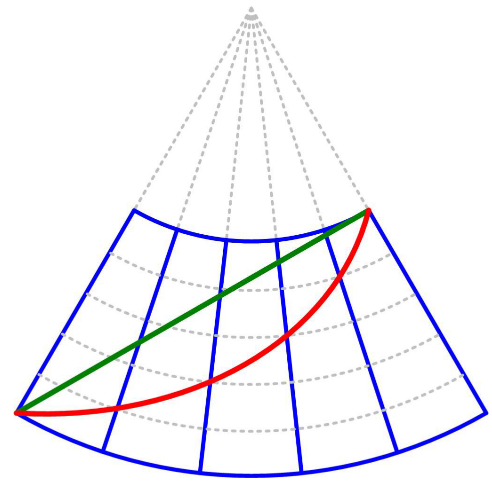

Figure 3

Then we see that the straight line from the bottom of the pillar to the top directly over the starting position turns out to be a leg of a 30-60 right triangle. Therefore, from the Pythagorean Theorem we conclude that its length is $15\sqrt{3}$ feet.

If you reimagine the cylinder wrapped up again, then the path of the streamer seems unintuitive, since it initially climbs rapidly and then wraps more shallowly as it approaches the top. A more intuitive path might be the one initially shown in the diagram for the problem, where the path cuts each "meridian" (vertical line to the apex) at the same angle (Figure 3). But like the constant-course direction on the sphere, it is not the shortest path.

Eureka Solution

The *Eureka* magazine gives the answer ($15\sqrt{3}$ ft.) and the suggestion "Develop the surface". If you didn't know what a developable surface was, you might find this cryptic.

References

[1] "The Problems Drive #9", *Eureka*, No. 12, October 1949. p.8