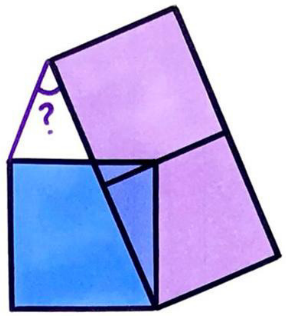
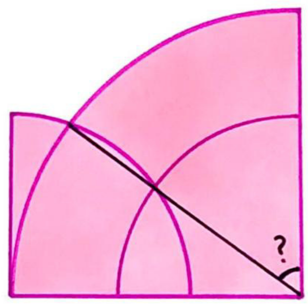
Geometric Puzzle Mystifiers

31 March 2023

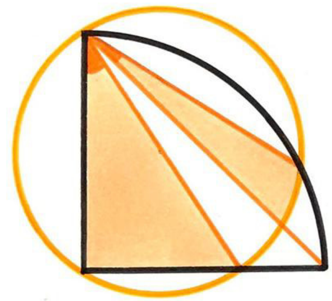
Jim Stevenson



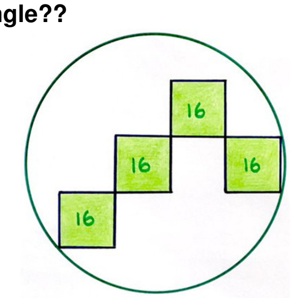
#1 3 Squares 1 Angle. Three squares. What's the angle??



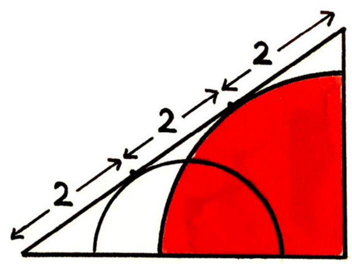
#2 Concentric Circles. **Two of these quarter circles are the same size. What's the angle?**



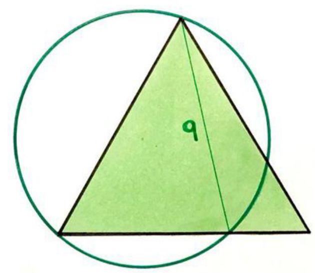
#3 Sun Rays. **What's the sum of the two marked angles?**



#4 Stair-Step Circle. **What's the area of the circle?**



#5 Downhill Slide. **What's the area of the quarter circle?**



#6 Circle Triangle. **The triangle is equilateral. What's the area of the circle?**

Here is yet another (belated) collection of beautiful geometric problems from Catriona Agg (née Shearer).

Solution to #1 3 Squares 1 Angle[1]

Since the tilt of the two stacked squares was arbitrary between 0° and 45°, we can consider extremes, where the angle θ approaches 45° in the limit. Figure 1 shows the case where the tilt is 0° and Figure 2 where the tilt is 45°. In both cases, θ is 45°, so we might anticipate it is also 45° for any intermediate tilt.

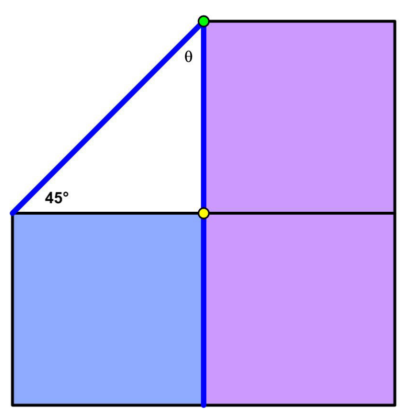


Figure 1

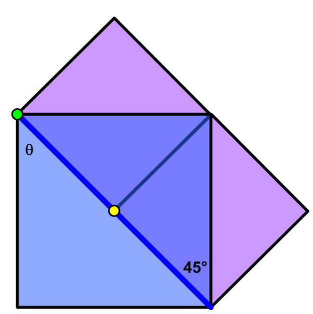


Figure 2

Figure 3 shows the general case for an intermediate tilt. The upper left corner of the lower square (yellow dot), being the vertex of a right triangle with fixed hypotenuse,

[1] 4 September 2021, https://twitter.com/Cshearer41/status/1434076663319498757

traces out one half of a semicircle between 45° and 90°. The upper left corner of the upper square (green dot) also traces out one half of a semicircle between 45° and 90°, only twice as far from the origin.

Therefore, we immediately see that θ is an inscribed angle of the larger circle whose central angle is always 90°. So θ must be half that value or 45°, which is what we wanted to find.

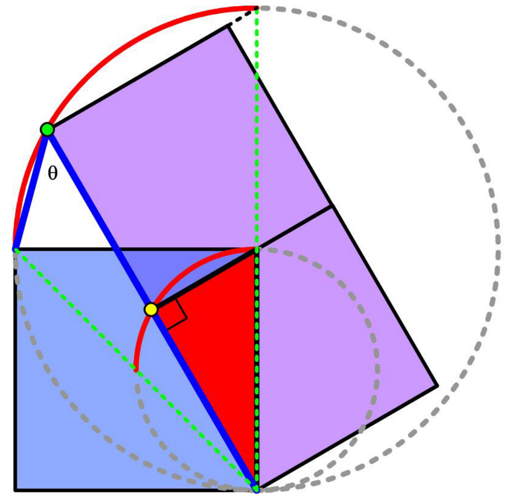


Figure 3

Solution to #2 Concentric Circles[2]

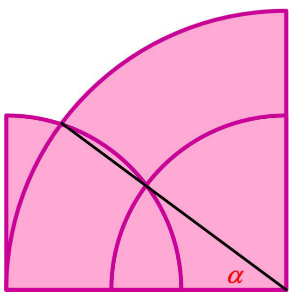


Figure 4

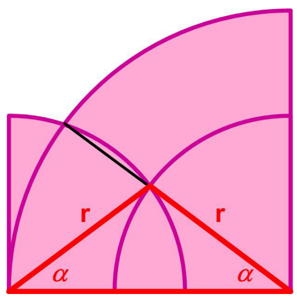


Figure 5

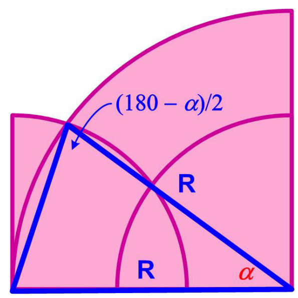


Figure 6

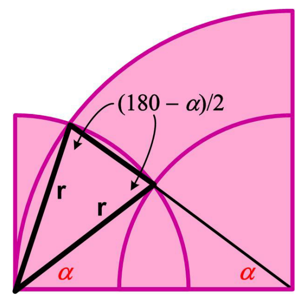


Figure 7

Let α be the complement to the unknown angle (Figure 4). Since the two small quarter circles intersect on the perpendicular bisector of the base line, the red triangle is isosceles and the two equal base angles are α (Figure 5). The large blue triangle is also isosceles with base angle (180° – α)/2 (Figure 6). And finally, the small black triangle is also isosceles with base the same base angle (180° – α)/2 (Figure 7). Therefore,

$$(180° - \alpha)/2 = 2\alpha \quad \Rightarrow \quad 5\alpha = 180° \quad \Rightarrow \quad \alpha = 36° \quad \Rightarrow \quad 90° - \alpha = 54°.$$

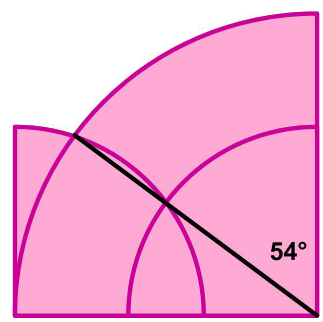


Figure 8

[2] 2 March 2021 https://twitter.com/Cshearer41/status/1366723349188067339

Solution to #3 Sun Rays[3]

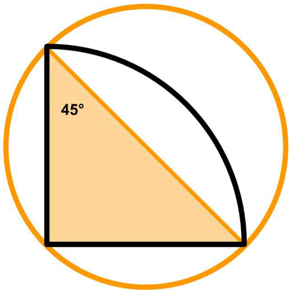


Figure 9

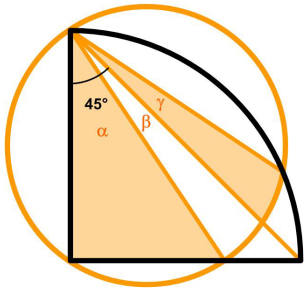


Figure 10

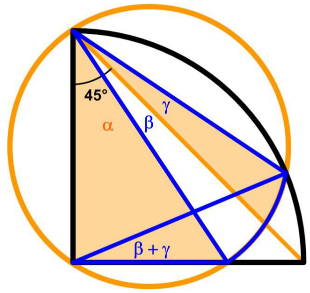


Figure 11

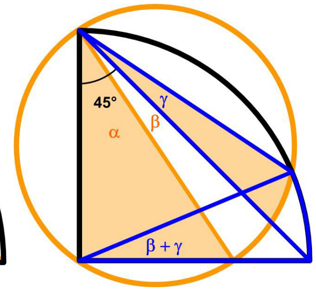


Figure 12

Notice that when the black quarter circle is completely inscribed in the circle, the sum of the two angles is 45° (Figure 9). Let α, β, and γ be the angles as shown in Figure 10. Then we are interested in the value of $\alpha + \gamma$, which we conjecture should be 45°. Notice that $\alpha + \beta = 45°$. Figure 11 shows the two equal inscribed angles $\beta + \gamma$ in the orange circle. Figure 12 shows that the inscribed angle γ in the quarter circle is half $\beta + \gamma$. Therefore,

$$(\beta + \gamma)/2 = \gamma \;\Rightarrow\; \beta = \gamma \;\Rightarrow\; 45° = \alpha + \beta = \alpha + \gamma.$$

Solution to #4 Stair-Step Circle[4]

Figure 13 provides the solution. Since the right-most green square touches the circle, the perpendicular bisector of the edge passes through the center of the circle and thus is a diameter. It also divides the edge of the square 4 into halves of size 2.

Because all the edges are parallel to one another as shown, the blue line from the upper right vertex of the right-most square to the lower left vertex of the left-most square passes through the mid-points of the edges of these two squares and the lower right edge of one of the other squares as shown.

The second blue line passes through this vertex and is perpendicular to the first blue line. It is easy to see this vertex is also the mid-point of the first blue line and so the second blue line passes through the center of the circle.

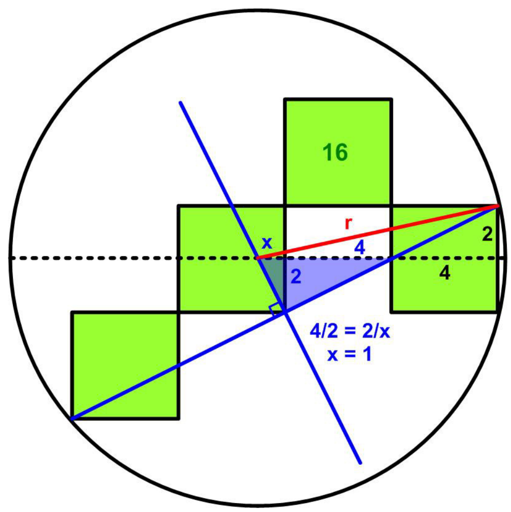


Figure 13

The red line of length r is drawn from the center to the upper right vertex of the right-most square and represents the radius of the circle. The blue right triangle is composed of two similar right triangles of sides 4 and 2, and 2 and x, respectively. Therefore they satisfy the geometric ratio 4/2 = 2/x, or x = 1. This gives the relation

$$r^2 = 2^2 + (1 + 4 + 4)^2 = 85$$

And so the area of the circle is 85π

[3] 2 April 2021 https://twitter.com/Cshearer41/status/1377937336399454208
[4] 6 April 2020 https://twitter.com/Cshearer41/status/1247073195829669893

Solution to #5 Downhill Slide[5]

This problem has a simple solution, as shown in Figure 14. Let R be the radius of the red quarter circle. Since the hypotenuse of the circumscribed right triangle is tangent to the quarter circle, the radius R is perpendicular and thus forms two similar sub-right-triangles. Therefore, by the geometric mean, we have $2/R = R/4$ or $R^2 = 8$. And so, the area of the quarter circle is $\frac{1}{4}\pi R^2 = 2\pi$.

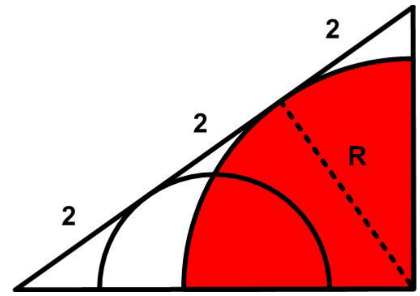


Figure 14

Solution to #6 Circle Triangle[6]

This is another problem with a simple solution, as shown in Figure 15. The orange lines are the radii of the circle and the angle between them is a central angle of 120°, because it subtends the same arc spanned by the 60° inscribed angle from the equilateral triangle.

Dropping the perpendicular bisector forms the orange 30-60 triangle so that $9/2 = \sqrt{3}r/2$ or $r = 3\sqrt{3}$. Therefore, the area of the circle is $\pi r^2 = 27\pi$.

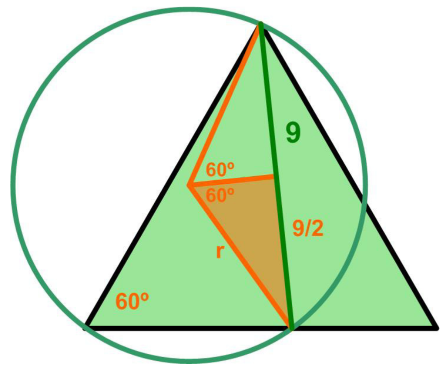


Figure 15

[5] 22 February 2022 https://twitter.com/Cshearer41/status/1496016925754892291
[6] 17 February 2021 https://twitter.com/Cshearer41/status/1361960717205860355