Moon Quarters Problem

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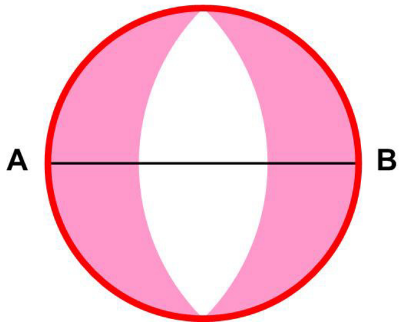

This is a straight-forward problem from the Scottish Mathematical Council Senior Mathematics Challenge ([1]).

A circle has radius 1 cm and *AB* is a diameter. Two circular arcs of equal radius are drawn with centres *A* and *B*. These arcs meet on the circle as shown. Calculate the shaded area.

There are several possible approaches and the SMC offers two examples.

My Solution

Figure 1 shows my decomposition of the circle into regions whose areas can be computed and used to obtain the desired answer. R is the radius of the large blue circle and r is the radius of the original (red) circle, where r = 1 cm. From the figure, we see that

$$R = \sqrt{2}\, r.$$

C is the area of the small red circle. *M* is the area of one of the quarter moons. S_1 is the area of the yellow sector of the large blue circle. *T* is the area of the green triangle bounded by the radii of the small red circle. S_2 is the sector of the small red circle bounded by the radii, and *L* is the difference in the areas of this sector and the triangle

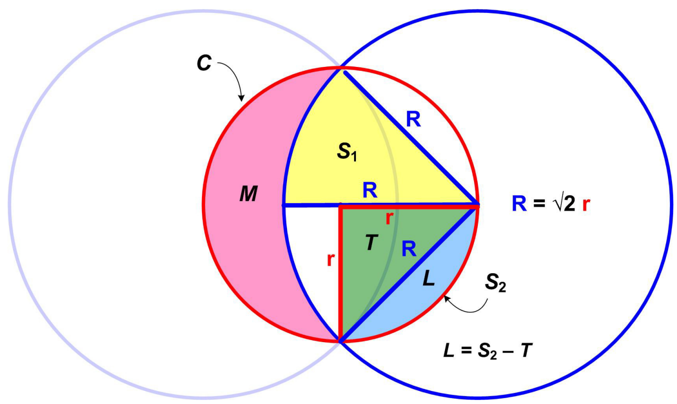

Figure 1

$$L = S_2 - T.$$

So we can construct the area for *M* as

$$\begin{aligned} M &= C - 2(S_1 + L) \\ &= C - 2(S_1 + S_2 - T) \\ &= C - 2S_1 - 2S_2 + 2T \\ &= \pi r^2 - \pi R^2/4 - \pi r^2/2 + r^2 \\ &= \pi(r^2 - 2r^2/4 - r^2/2) + r^2 \\ &= r^2 \end{aligned}$$

So, since r = 1 cm, the area of two quarter moons is

$$2M = 2r^2 = 2\ \text{cm}^2$$

SMC Solutions

SMC provided two solutions ([2]).

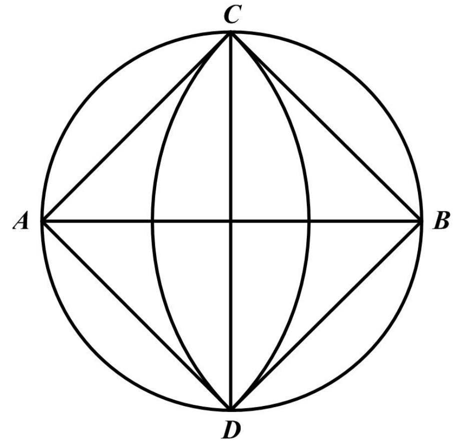

Figure 2

Solution 1.

$$BC^2 = 1^2 + 1^2 = 2$$

$\angle CBA = 45°$, so $\angle CAD = 90°$. The radius of the sector *CBD* is $BC = \sqrt{2}$, so the area of sector $CBD = \pi BC^2/4 = \pi/2$. $CD = 2$, so the area of $\Delta BCD = ½ \cdot 2 \cdot 1 = 1$. Therefore the unshaded area is $2(\pi/2 - 1) = \pi - 2$. But the area of the full circle is $\pi \cdot 1^2 = \pi$. So the shaded region is

$$\pi - (\pi - 2) = 2 \text{ cm}^2$$

Solution 2.

Let *C* and *D* be the points shown in the diagram [Figure 2]. Then $AC^2 = 1^2 + 1^2 = 2$, so that the radius of the circular arc with centre *A* is $\sqrt{2}$. Then the shaded area is

(area of circle of radius 1) – 2(area of the segment of a circle, radius $\sqrt{2}$, subtended by an angle $\pi/2$)

= π – 2(area of the sector of a circle, radius $\sqrt{2}$, subtended by an angle $\pi/2$ – area of ΔACD)

= π – 2(quarter of area of the sector of circle of radius $\sqrt{2}$ – area of ΔACD)

$= \pi - 2(¼ \cdot 2\pi - ½ \cdot \sqrt{2} \cdot \sqrt{2})$

$= \pi - 2(\pi/2 - 1) = 2 \text{ cm}^2$

References

[1] "Senior Division: Problems 1 S3" *Mathematical Challenge 2011−2012*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-1112-Q1.pdf)

[2] "Senior Division: Problems 1 Solutions S3" *Mathematical Challenge 2011−2012*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-1112-S1.pdf)