Linked Triangles Problem

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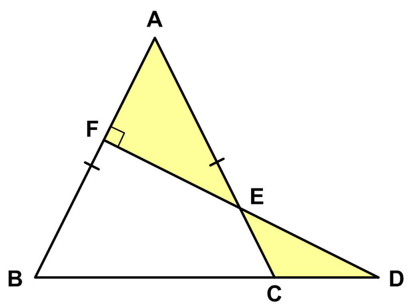

I found this problem from the 1981 Canadian Math Society's magazine, *Crux Mathematicorum* ([1]), to be quite challenging.

Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, China.

An isosceles triangle has vertex A and base BC. Through a point F on AB, a perpendicular to AB is drawn to meet AC in E and BC produced in D. Prove synthetically that

Area of AFE = 2 Area of CDE   if and only if AF = CD.

My Solution

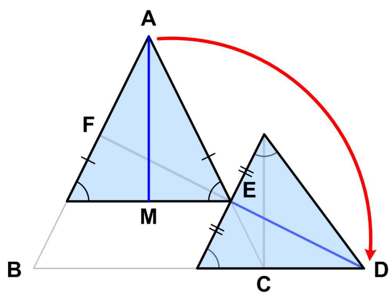

Figure 1

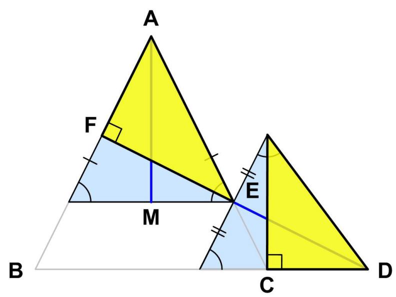

Figure 2

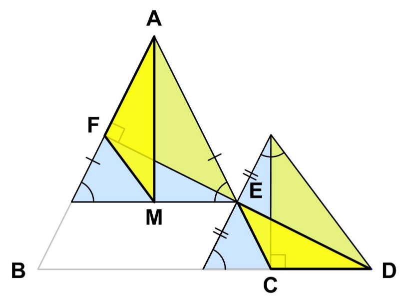

Figure 3

Rather than show a final figure with a number of overlapping and confusing triangles, I thought I would build the result in steps.

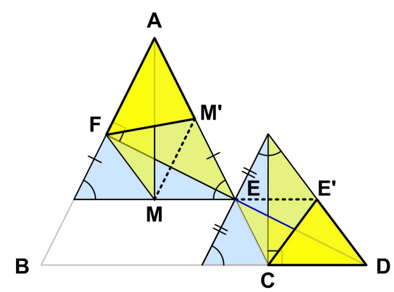

Figure 4

Step 1 (Figure 1). Draw a line through E parallel to BD to form the base of a (blue) isosceles triangle. Drop the perpendicular bisector (altitude) from A to the midpoint M. Now construct a second (blue) isosceles triangle with midpoint E on the base and altitude ED. The base through E is chosen parallel to line AB so that the second isosceles triangle is similar to the first, since all the angles are the same. In effect, it is a rotated and possibly shrunk or expanded version of the first triangle.

Step 2 (Figure 2). Add the (yellow) right triangle AFE from the problem to the first blue triangle, and add a second right triangle to the second blue triangle in the same position corresponding to the first. These two right triangles are also similar since their angles are the same.

Step 3 (Figure 3). Now add the second (yellow) triangle ECD from the problem to the *second* blue triangle, and then add its corresponding version to the first blue triangle. Again because they have the same angles, they are similar.

Step 4 (Figure 4). Finally, add a (dashed) line to the first blue triangle parallel to AB from the midpoint M to the midpoint M' on line AE, and add a similar (dashed) line to the second blue triangle parallel to BD from the midpoint E to the midpoint E'. Then the area of triangle AFM is equal to the area of (sheared) triangle AFM', which is half the area of the right triangle AFE (same altitude, half the base). Similarly, the area of triangle CDE is the same as the area of triangle CDE', which is half the area of the right triangle in the second blue triangle. As with all the other triangles, AFM' is similar to CDE'.

Therefore, the area of triangle AFE (in the first blue triangle) is twice the area of triangle CDE (in the second blue triangle) if and only if triangles AFM' and CDE' are congruent, which is true if and only if AF = CD.

Comment. It took me several days to find the appropriate transformations and formulate the argument. It was quite tricky to find a way to link the two (yellow) triangles in the original problem.

Crux Mathematicorum Solution

Their solution is essentially the same as mine, based on similar triangles.

Solution by E.C. Buissant des Amorie, Amstelveen, The Netherlands.

If a line through E parallel to BC meets the altitude issued from A in G, then the points A, F, G, E are concyclic, and it is easy to verify that certain angles are properly designated as congruent in the figure on the next page [Figure 5], as a consequence of which

triangles CDE and FAG are (inversely) similar. (1)

Since clearly [AFE][1] = 2[AFG],[2] we have, from (1),

$$[AFE] = 2[CDE] \iff [AFG] = [CDE] \iff AF = CD.$$

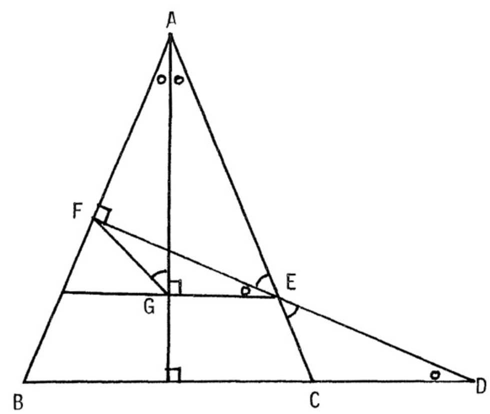

Figure 5

Also solved by JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; GEORGE TSINTSIFAS, Thessaloniki, Greece; DAVID ZAGORSKI, student, Massachusetts Institute of Technology; and the proposer,

Editor's comment.

The proposal asked for a synthetic solution, and the one we gave above was by far the best. Five of the other solutions used "synthetic" trigonometry.

References

[1] Tan, Kaiday, "Problem 662," *Crux Mathematicorum*, Vol.7 No.7 August, Canadian Mathematical Society, 1981. p.204.

[2] Buissant, E.C., "Problem 662 Solution," *Crux Mathematicorum*, Vol.8 No.7 August, Canadian Mathematical Society, 1982. pp.218-9.

[1] JOS: "[ ]" signifies area.
[2] JOS: Perhaps the shearing argument in Step 4 of my solution would clarify this.