Log Lunacy

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This is an initially mind-boggling problem from the 1995 American Invitational Mathematics Exam (AIME) ([1]).

> Find the last three digits of the product of the positive roots of
>
> $$\sqrt{1995}\, x^{\log_{1995} x} = x^2$$

My Solution

Take $\log_{1995}$ of both sides and simplify.

$$\log_{1995} (1995^{½}\, x^{\log_{1995} x}) = \log_{1995} x^2$$

$$½ \log_{1995} 1995 + (\log_{1995} x)^2 = 2\log_{1995} x$$

Setting $y = \log_{1995} x$, we get

$$y^2 - 2\,y + ½ = 0$$

and so

$$y = (2 \pm \sqrt{(4 - 2)})/2 = 1 \pm ½ \sqrt{2}$$

Therefore the two positive roots of the original equation, via $x = 1995^y$, are

$$x = 1995^{\,1 + ½ \sqrt{2}} \quad \text{and} \quad x = 1995^{\,1 - ½ \sqrt{2}}$$

So the product is

$$(1995^{\,1 + ½ \sqrt{2}})\,(1995^{\,1 - ½ \sqrt{2}}) = 1995^2 = 3980025,$$

which means the last three digits of the result are 025.

AIME Solutions

AIME's first solution is the same as mine, only they had a slicker way of obtaining the final digits without using a calculator.

Solution 1

Taking the $\log_{1995}$ (logarithm) of both sides and then moving to one side yields the quadratic equation $2(\log_{1995} x)^2 - 4(\log_{1995} x) + 1 = 0$. Applying the quadratic formula yields that $\log_{1995} x = 1 \pm \frac{\sqrt{2}}{2}$. Thus, the product of the two roots (both of which are positive) is $1995^{1+\sqrt{2}/2} \cdot 1995^{1-\sqrt{2}/2} = 1995^2$, making the solution $(2000-5)^2 \equiv \boxed{025} \pmod{1000}$.

Solution 2

Instead of taking $\log_{1995}$, we take $\log_x$ of both sides and simplify:

$\log_x(\sqrt{1995}x^{\log_{1995} x}) = \log_x(x^2)$

$\log_x \sqrt{1995} + \log_x x^{\log_{1995} x} = 2$

$\frac{1}{2}\log_x 1995 + \log_{1995} x = 2$

We know that $\log_x 1995$ and $\log_{1995} x$ are reciprocals, so let $a = \log_{1995} x$. Then we have $\frac{1}{2}\left(\frac{1}{a}\right) + a = 2$. Multiplying by $2a$ and simplifying gives us $2a^2 - 4a + 1 = 0$, as shown above.

Because $a = \log_{1995} x$, $x = 1995^a$. By the quadratic formula, the two roots of our equation are $a = \frac{2 \pm \sqrt{2}}{2}$. This means our two roots in terms of x are $1995^{\frac{2+\sqrt{2}}{2}}$ and $1995^{\frac{2-\sqrt{2}}{2}}$. Multiplying these gives 1995^2

$1995^2 \pmod{1000} \equiv 995^2 \pmod{1000} \equiv (-5)^2 \pmod{1000} \equiv 25 \pmod{1000}$, so our answer is $\boxed{025}$.

Solution 3

Let $y = \log_{1995} x$. Rewriting the equation in terms of y, we have

$$\sqrt{1995}\left(1995^y\right)^y = 1995^{2y}$$

$$1995^{y^2+\frac{1}{2}} = 1995^{2y}$$

$$y^2 + \frac{1}{2} = 2y$$

$$2y^2 - 4y + 1 = 0$$

$$y = \frac{4 \pm \sqrt{16-(4)(2)(1)}}{4} = \frac{4 \pm \sqrt{8}}{4} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Thus, the product of the positive roots is $\left(1995^{\frac{2+\sqrt{8}}{2}}\right)\left(1995^{\frac{2-\sqrt{8}}{2}}\right) = 1995^2 = (2000-5)^2$, so the last three digits are $\boxed{025}$.

References

[1] "Problem 2" 1995 AIME Problems (https://artofproblemsolving.com/wiki/index.php/1995_AIME_Problems)