Incredible Trick Puzzle

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Jim Stevenson

Here is another typical sum puzzle from Presh Talwalkar ([1]).

Solve the following sums:

1/(1×3) + 1/(3×5) + 1/(5×7) + 1/(7×9) + 1/(9×11) =

1/(4×7) + 1/(7×10) + 1/(10×13) + 1/(13×16) =

1/(2×7) + 1/(7×12) + 1/(12×17) + … =

The only reason I am including this puzzle is that Talwalkar gets very excited about deriving a formula that can solve sums of this type. This gives me an opportunity to discuss the “formula vs. procedure” way of doing math.

My Solution

The key is to recognize that these sums are a type of telescoping sum. Writing the first sum in a straight telescoping form would yield:

$$\left(\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}\right)=\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)\right]$$

or

$$\frac{1}{2}\left(1-\frac{1}{11}\right)=\frac{5}{11}$$

The second sum is simply

$$\left(\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\frac{1}{10\cdot 13}+\frac{1}{13\cdot 16}\right)=\frac{1}{3}\left[\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\left(\frac{1}{10}-\frac{1}{13}\right)+\left(\frac{1}{13}-\frac{1}{16}\right)\right]=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{16}\right)=\frac{1}{16}$$

The last, infinite, sum is just

$$\left(\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\frac{1}{12\cdot 17}+\ldots\right)=\frac{1}{5}\left[\left(\frac{1}{2}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{12}\right)+\left(\frac{1}{12}-\frac{1}{17}\right)+\ldots\right]=\frac{1}{5}\left(\frac{1}{2}\right)=\frac{1}{10}$$

So to solve these problems we just apply the same procedure with slight tweaks to adjust for specific differences.

Talwalkar’s Solution (with Comment)

Talwalkar’s approach is to derive a general formula that would solve all these types of problems at once. The derivation of the formula is basically a repetition of the procedure we employed above.

I learned an incredible trick to solve these kinds of problems from Study IQ Maths on TikTok.[1] In the solution section, I will explain the trick and give a proof of why it works.

[1] https://www.tiktok.com/@studyiq_maths/video/6745984377672584449

Answer To Incredible Trick To Solve These Sums

Let's solve the first sum.

$$1/(1\times3) + 1/(3\times5) + 1/(5\times7) + 1/(7\times9) + 1/(9\times11) =$$

Here's the technique. Count the number of terms in the sum—there are 5. Then take the first factor in the first denominator (1) and multiply it by the largest factor in the last denominator (11). Divide the number of terms by the product and you have the answer:

(number of terms)/(first factor × last factor) = $5/(1\times11) = 5/11$

That's it! Let's use the technique on the second sum.

$$1/(4\times7) + 1/(7\times10) + 1/(10\times13) + 1/(13\times16) =$$

There are 4 terms in the sum, and then we have the first factor is 4 and the last factor is 16. Thus the sum is:

(number of terms)/(first factor × last factor) = $4/(4\times16) = 1/16$

Wow! Now let's solve the infinite series:

$$1/(2\times7) + 1/(7\times12) + 1/(12\times17) + \ldots =$$

We can't simply count the number of terms because the number of terms is infinite. But we can modify the formula. The difference in the factors is 5. Multiply this by the first factor, 2, and take the reciprocal of the product. The sum is:

1/(first factor × difference of factors) = $1/(2\times5) = 1/10$

This is an incredible technique! But why does it work?

Proof

The general form of such sums is:

$$1/(a \times (a + d)) + 1/((a + d)\times (a + 2d)) + \ldots + 1/((a + (n - 1)d)\times (a + nd))$$

A general term in the series has the form:

$$1/(k \times (k + d))$$

We will first use the method of partial fractions:

$$1/(k \times (k + d)) = P/k + Q/(k + d)$$

$$1 = P(k + d) + Qk$$

$$1 = Pd + Pk + Qk$$

$$1 = Pd + k(P + Q)$$

Equating the constant term gives $P = 1/d$. Since the coefficient on n is 0 on the left hand side, we then have:

$$P + Q = 0$$

$$1/d + Q = 0$$

$$Q = -1/d$$

Thus we have:

$$1/(k \times (k + d)) = (1/d)/k - (1/d)/(k + d)$$

We can apply the formula to each term of the partial sum:

$$S_n = 1/(a \times (a + d)) + 1/((a + d)\times (a + 2d)) + \ldots + 1/((a + (n - 1)d)\times (a + nd))$$

$$S_n = (1/d)/a - (1/d)/(a + d) + (1/d)/(a + d) - (1/d)/(a + 2d) + (1/d)/(a + 2d) - (1/d)/(a + 3d) + \ldots + (1/d)/(a + (n - 1)d) - (1/d)/(a + nd)$$

We can simplify this sum tremendously because it is a telescoping sum. Notice the second term with the third term, the fourth term cancels with the fifth terms, and so on, until the penultimate term which also cancels with its previous term. Only the first and last terms remain, giving:

$$\begin{aligned} S_k &= (1/d)/a - (1/d)/(a + nd) \\ &= 1/da - 1/(d(a + nd)) \\ &= (a + nd - a)/(da(a + nd)) \\ &= nd/(da(a + nd)) \\ &= n/(a(a + nd)) \\ &= \textbf{(number of terms)/(first factor} \times \textbf{last factor)} \end{aligned}$$

This is precisely the formula we wanted to prove. For the infinite series, we will take the sum as n goes to infinity. First distribute in the denominator to get:

$$n/(a(a + nd)) = n/(a^2 + nad)$$

The limit as n goes to infinity is based on the factors involving n. So the limit tends to the fraction $n/(nad) = 1/(ad)$. Thus the infinite sum is $1/(ad)$. This is the formula we wished to prove because the numerator is 1, and the denominator is the product of the first factor a and the difference between factors d.

Now we have formally justified why these formulas work. So go forth and use these mind-blowing formulas to solve all such challenging sums in your head! I'm sure they will come in handy for an exam or a competition!

Reference

Study IQ Maths on TikTok (https://www.tiktok.com/@studyiq_maths/video/6745984377672584449)

Comment. I have mentioned in the past that as I got older, I could not remember formulas (since I rarely used them), but somehow I remembered procedures that would derive the formulas. Some of the procedures could be shortened if I remembered some basic seed formulas, like $\sin^2 + \cos^2 = 1$, area of circle = πr^2, or volume of sphere = ${}^4/_3\pi r^3$. Notice you can derive the surface area of the sphere as the derivative of the volume, namely, $4\pi r^2$, and the circumference of a circle from the derivative of its area: $2\pi r$. In other words the change in volume or change in area is determined by how much flows through the boundary. Some things are truly forgotten such as the distance of the directrix from the corresponding conic section or the meaning of the latus rectum. For conics, it takes me a while to remember or derive the relation between the semimajor, semiminor, and center (a, b, c, respectively). Since c is the distance from the center to the focus of an ellipse, and the distance from the foci to semiminor axis b is 2a, from the Pythagorean Theorem (which I never forget), we get $a^2 = b^2 + c^2$. Similar thoughts get me the relation for hyperbolas.

So, I don't see any advantage in learning a formula for something that is easily derived when needed. Now it is fascinating that the difference in factors in the denominator disappears for finite sums, but remains for infinite sums.

References

[1] Talwalkar, Presh, “Incredible Trick To Solve These Sums” *Mind Your Decisions*, 15 February 2022 (https://mindyourdecisions.com/blog/2022/02/15/incredible-trick-to-solve-these-sums/)