“Fermat’s Last Theorem” Puzzle

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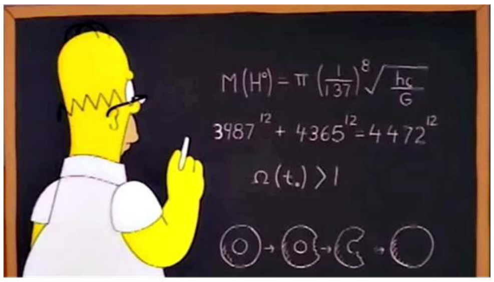

Numberphile/Youtube

Here is a mind-numbing logic puzzle from *Futility Closet* ([1]).

A puzzle by H.A. Thurston, from the April 1947 issue of *Eureka* ([2]), the journal of recreational mathematics published at Cambridge University:

FIVE people make the following statements:—

(1) *Either* (a) 3’s statement is false and 4’s is true.
Or (b) 2’s and 5’s are both false.
(2) *Either* 4’s statement is false *or* 3’s is false.
Or 1’s and 5’s are both false.
(3) 2’s statement is true *or* 4’s and 5’s are both true.
Moreover, *either* 5’s is false *or* 4’s is true.
(4) 3’s statement is false *or* 1’s is true.
(5) Fermat’s last theorem is true.

Which of these statements are true and which false? It will be found on trial that there is only one possibility. Thus, prove or disprove Fermat’s last theorem.

Normally I would forgo something this complicated, but I thought I would give it a try. I was surprised that I was able to solve it, though it took some tedious work. (Hint: truth tables. See the “Pointing Fingers”[1] post regarding truth tables.)

One important note. The author is a bit cavalier about the use of “Either …, or …”. In common parlance this means “either P is true or Q is true, *but not both*” (exclusive “or”: XOR), whereas in logic “or” means “either P is true or Q is true, *or possibly both*” (inclusive “or”: OR). I assumed all “Either …, or …” and “or” expressions were the logical inclusive “or”, which turned out to be the case.

My Solution

First, I replaced the five numbered statements with the letters A, B, C, D, E and translated the statements into symbolic logic, as shown in Table 1.

Table 1

1.	A. Either (C is false and D is true) Or (B is false and E is false)	$A \equiv (\sim C \wedge D) \vee (\sim B \wedge \sim E)$
2.	B. Either (D is false or C is false) Or (A is false and E is false)	$B \equiv (\sim D \vee \sim C) \vee (\sim A \wedge \sim E)$
3.	C. (B is true Or (D is true and E is true)) And (Either E is false or D is true)	$C \equiv (B \vee (D \wedge E)) \wedge (\sim E \vee D)$
4.	D. C is false or A is true	$D \equiv \sim C \vee A$
5.	E. Fermat’s Last Theorem is true	E

[1] http://josmfs.net/2020/09/19/pointing-fingers/

Then I built a monster truth table. Since there are 5 statements, there are $2^5 = 32$ choices for combinations of T and F for the 5 letters. I then added columns for the symbolic logic translations of the statement and subdivided them into three subcolumns to build up the truth or falsity of the statements (see Table 2).

I began with the first statement A in the first column and computed its truth value based on all the values for B, C, D, E. If the resulting truth value for a computation disagreed with the chosen truth value for A in that instance, this represented a contradiction and I discarded the "row" by filling it with red. Wherever there was an agreement with the computed A and the chosen A, I then had to look at other statements. I chose ones that involved more "ands", since these were quicker to eliminate. I decided to look at statement C next. It eliminated more cases. For those that were not contradicted, I then considered the D statement. The result of all these computations was that all but two rows were eliminated. I then checked statement B and found it eliminated one of the remaining two. That left just one consistent result, which is highlighted in Table 2 and given in Table 3.

Table 2

A	B	C	D	E	A (~C∧D) ∨ (~B∧~E)			B (~D∨~C) ∨ (~A∧~E)			C (B∨(D∧E)) ∧ (~E∨D)			D ~C∨A			E
T	T	T	T	T	F	F	F										
T	T	T	T	F	F	F	F										
T	T	T	F	T	F	F	F										
T	T	T	F	F	F	F	F										
T	T	F	T	T	T	T	F				T	T	T				
T	T	F	T	F	T	T	F				T	T	T				
T	T	F	F	T	F	F	F										
T	T	F	F	F	F	F	F										
T	F	T	T	T	F	F	F										
T	F	T	T	F	F	T	T				F	F	T				
T	F	T	F	T	F	F	F										
T	F	T	F	F	F	T	T				F	F	T				
T	F	F	T	T	T	T	F				T	T	T				
T	F	F	T	F	T	T	T	T	T	F	F	F	T	T	T	T	
T	F	F	F	T	F	F	F										
T	F	F	F	F	F	F	T										
F	T	T	T	T	F	F	F				T	T	T	F	F	F	
F	T	T	T	F	F	F	F				T	T	T	F	F	F	
F	T	T	F	T	F	F	F				T	F	F				
F	T	T	F	F	F	F	F	T	T	T	T	T	T	F	F	F	F
F	T	F	T	T	T	T	F										
F	T	F	T	F	T	T	F										
F	T	F	F	T	F	F	F				T	F	F	T	T	F	
F	T	F	F	F	F	F	F				T	T	T				
F	F	T	T	T	F	F	F				T	T	T	F	F	F	
F	F	T	T	F	F	T	T										
F	F	T	F	T	F	F	F				F	F	F				
F	F	T	F	F	F	T	T										
F	F	F	T	T	T	T	F										
F	F	F	T	F	T	T	T										
F	F	F	F	T	F	F	F				F	F	F	T	T	F	
F	F	F	F	F	F	T	T										

Table 3

1.	A. Either (C is false and D is true) Or (B is false and E is false)	False
2.	B. Either (D is false or C is false) Or (A is false and E is false)	True
3.	C. (B is true Or (D is true and E is true)) And (Either E is false or D is true)	True
4.	D. C is false or A is true	False
5.	E. Fermat's Last Theorem is true	False

Futility Closet Solution

The *Futility Closet* answer agreed with mine, but as they said, there was no solution provided.

The answer appears in the March 1948 issue ([3]):

"The statements of 2 and 3 are true, those of 1, 4 and 5 are false."

But no full solution is given.

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Comment

I can't help emphasizing the power of mathematics here. We have yet another instance of translating a "real world" problem into a mathematical model (machine)—this time a truth table—"turning the crank", that is, performing the associated operations of the "machine", producing a result, and then translating it back to the "real world." This is the excitement, pleasure, and wonder of mathematics. Furthermore, is the machine invented or is it already inherent in the real world awaiting to be discovered?

References

[1] "Fermat's Last Theorem", *Futility Closet*, 23 June 2021 (https://www.futilitycloset.com/2021/06/23/fermats-last-theorem/)

[2] Thurston, H.A., "Fermat's Last Theorem" (Problem), *Eureka*, No. 9, April 1947. p.22 (https://www.archim.org.uk/eureka/archive/Eureka-9.pdf)

[3] ———, "Fermat's Last Theorem" (Solution), *Eureka*, No.10, March 1948. p.25 (https://www.archim.org.uk/eureka/archive/Eureka-10.pdf)

[2] https://creativecommons.org/licenses/by/4.0/