Fireworks Rocket

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This is another physics-based problem from Colin Hughes's *Maths Challenge* website (mathschallenge.net) ([1]) that may take a bit more thought.

A firework rocket is fired vertically upwards with a constant acceleration of 4 m/s^2 until the chemical fuel expires. Its ascent is then slowed by gravity until it reaches a maximum height of 138 metres.

Assuming no air resistance and taking g = 9.8 m/s^2, how long does it take to reach its maximum height?

I can never remember the formulas relating acceleration, velocity, and distance, so I always derive them via integration.

My Solution

Figure 1 shows a space-time diagram of the problem setup. Initially, the rocket takes off from standing still, accelerates at 4 m/s^2, and arrives at a height h_0 and vertical velocity v_0 at time t_0 when the fuel runs out and the rocket shuts off. The rocket coasts upwards under a downward acceleration of 9.8 m/s^2 from gravity, until it stops moving at a height of 138 m at a time t_{max}.

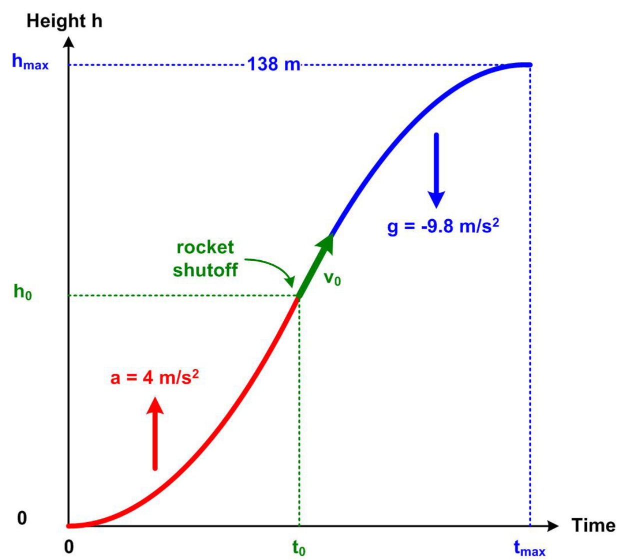

Figure 1

During powered flight we have the rocket's velocity and height are given by the integrals

$$v(t) = \int_0^t 4dt = 4t$$

and

$$h(t) = \int_0^t v(t)dt = \int_0^t 4tdt = 2t^2$$

Therefore,

$$v_0 = v(t_0) = 4t_0 \quad \text{and} \quad h_0 = h(t_0) = 2t_0^2. \qquad (*)$$

During unpowered flight we have the rocket's velocity and height are given by the integrals

$$v(t) = v_0 + \int_{t_0}^t (-9.8)dt = v_0 - 9.8(t - t_0)$$

and

$$h(t) = h_0 + \int_{t_0}^t v(t)dt = h_0 + \int_{t_0}^t (v_0 - 9.8(t - t_0))dt = h_0 + \left[v_0(t - t_0) - \frac{9.8(t - t_0)^2}{2} \right]$$

Therefore, from (*)

$$0 = v(t_{max}) = v_0 - 9.8(t_{max} - t_0) \quad \Rightarrow \quad v_0 = 9.8(t_{max} - t_0) = 4t_0 \qquad (**)$$

and

$$138 = h(t_{max}) = h_0 + v_0(t_{max} - t_0) - 9.8(t_{max} - t_0)^2/2.$$

So, from (*) and (**)

$$138 = 2t_0^2 + 9.8(t_{max} - t_0)^2 - 9.8(t_{max} - t_0)^2/2$$
$$= 2t_0^2 + 9.8(t_{max} - t_0)^2/2$$
$$= 2t_0^2 + 9.8(4t_0)^2/(2 \cdot 9.8^2)$$
$$= 2t_0^2(13.8/9.8)$$

Now from (**),

$$t_0 = t_{max}\,(9.8/13.8),$$

so

$$138 = 2(9.8/13.8)^2(13.8/9.8)\,t_{max}^2 = 2(7 \cdot 14/138)\,t_{max}^2$$

and so

$$t_{max}^2 = (138/14)^2 = (69/7)^2$$

implies

$$t_{max} = 69/7 \approx 9.863 \text{ seconds}$$

Maths Challenge Solution

The following is the *Maths Challenge* solution verbatim with some minor reformatting and paragraph breaks for clarity. No diagram was provided, so the definitions of the indexed values had to be inferred: index 1 denotes when the rocket's fuel runs out, index 2 when its maximum height is reached. *u* seems to be an initial value for the velocity over each interval. *v* seems to be the value of the velocity at the end of each interval. *w* seems to be the value of the velocity at the end of the first interval.

During acceleration phase (as fuel burns), $a = 4$, $u = 0$, let $v = w$.

$$v = u + at \;\Rightarrow\; w = 4t_1 \quad (1)$$

$$v^2 = u^2 + 2as\text{ [1]} \;\Rightarrow\; w^2 = 8s_1 \quad (2)$$

During deceleration phase (fuel expired), $a = -9.8$, $u = w$, $v = 0$.

$$v = u + at \;\Rightarrow\; 0 = w - 9.8t_2 \;\Rightarrow\; w = 9.8t_2 \quad (3)$$

$$v^2 = u^2 + 2as \;\Rightarrow\; 0 = w^2 - 19.6s_2 \;\Rightarrow\; w^2 = 19.6s_2 \quad (4)$$

Equating (1) and (3), $4t_1 = 9.8t_2 \;\Rightarrow\; t_2 = (20/49)t_1$,

so total time to reach maximum height,

$$t_1 + t_2 = (69/49)t_1.$$

Equating (2) and (4), $8s_1 = 19.6s_2 \;\Rightarrow\; s_2 = (20/49)s_1$,

and as $s_1 + s_2 = (69/49)s_1 = 138$,

we get $s_1 = 98$.

[1] JOS: I never remember this formula. If $s = v_0t + at^2/2$ and $v = v_0 + at$, then $v^2 = v_0^2 + 2v_0at + a^2t^2 = v_0^2 + 2as$.

Using $s = ut + ½at^2$ during acceleration phase,

$$s_1 = 2t_1^2,\ \ t_1 = \sqrt{(s_1/2)} = 7.$$

Hence time to reach maximum height is (69/49)·7 = 69/7 seconds.

References

[1] Hughes, Colin, "Firework Rocket", *Maths Challenge*, (mathschallenge.net) (March 2004) #162 p.47. Difficulty: 4 Star. "A four-star problem: A comprehensive knowledge of school mathematics and advanced mathematical tools will be required."