Triangle Stripes Problem

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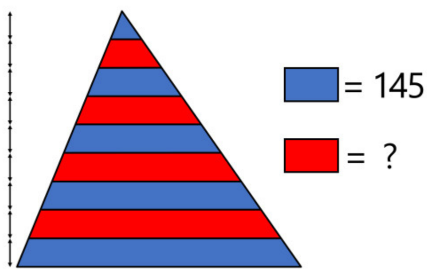
11 November 2022



This is a fairly straight-forward problem from Presh Talwalkar ([1]).

A triangle is divided by 8 parallel lines that are equally spaced, as shown below. Starting from the top small triangle, color each alternate stripe in blue and color the remaining stripes in red. If the blue stripes have a total area of 145, what is the total area of the red stripes?

My Solution

Let A be the area of the entire triangle, A_B the area of the blue lines, and A_R the area of the red lines, to be determined. Shear the triangle horizontally to the left so that the left hand side of the triangle is perpendicular. Because the motion is horizontal all the areas of the cascading similar triangles remain unchanged (the altitudes remain the same, as well as the bases).

Since equally-spaced parallel lines cut off equally-spaced line segments on the edges of the triangle, we can drop a set of equally-spaced lines parallel to the left edge, and thus perpendicular to the base (Figure 1). Let the spacing between the horizontal parallels be Δh, and the spacing between the vertical parallels be Δb. Then we have the following relations:

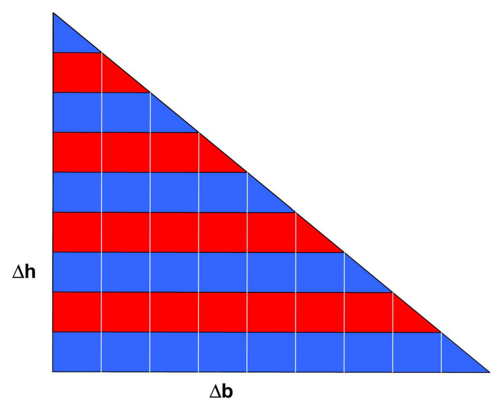


Figure 1

$$
\begin{aligned}
A_B &= 145 = [(2 + 4 + 6 + 8) + 5/2]\Delta b\Delta h = (45/2)\, \Delta b\Delta h \\
A_R &= A - A_B \\
&= 9\Delta b\; 9\Delta h/2 - 145 \\
&= 81(2\cdot 145/45)/2 - 145 \qquad \text{(eliminating } \Delta b\Delta h\text{)} \\
&= (81/45 - 1)145 \\
&= (9/5 - 1)145 \\
&= (4/5)145 \\
&= 4\cdot 29 \\
&= 116
\end{aligned}
$$

Talwalkar Solution

We can solve the problem in two ways.

Method 1: similar triangles

Recall: if two similar triangles have corresponding sides in a ratio of x, then their areas will be in a ratio of x^2.

We can form a triangle between the top vertex and the two endpoints of one of the parallel lines. Starting from the top parallel line, label these triangles as T_1, T_2, …, T_9. Because the horizontal lines are parallel, all of the triangles are similar. Supposing the top triangle has a height of h from the top vertex to the horizontal base, the other triangles have heights of $2h$, …, $9h$, respectively.

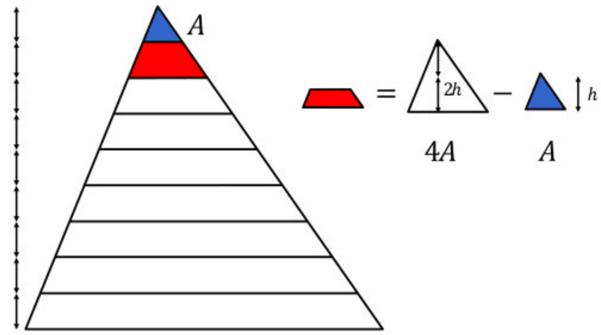


Figure 2

Suppose the top blue triangle, T_1, has an area equal to A. Then the area of the top red stripe is equal to:

$$\begin{aligned}\text{area}(T_2) - \text{area}(T_1) &= A(2^2) - A \\ &= 4A - A \\ &= 3A\end{aligned}$$

The area of each subsequent stripe can be calculated in a similar manner:

$$\begin{aligned}\text{area}(T_{k+1}) - \text{area}(T_k) &= A(k+1)^2 - A(k)^2 \\ &= A(k^2 + (2k+1)) - Ak^2 \\ &= Ak^2 + A(2k+1) - Ak^2 \\ &= A(2k+1)\end{aligned}$$

Thus each subsequent stripe has an area $2A$ more than the previous one, and the areas of the stripes are A, $3A$, …, $17A$. The blue stripes have an area equal to 145. Every other stripe starting from the top is blue, so we have:

$$A + 5A + 9A + 13A + 17A = 145$$

$$45A = 145$$

$$A = 29/9$$

The red stripes then have an area equal to:

$$\begin{aligned}3A + 7A + 11A + 15A &= 36A \\ &= 36(29/9) \\ &= 116\end{aligned}$$

Method 2: a nice tiling

After I worked out the algebraic solution above, I realized a nice geometric pattern: The large triangle can actually be partitioned by the smallest triangle at the top! Supposing the top triangle has an area equal to A, it is only a matter of counting to see the stripes have areas of $3A$, $5A$, …, $17A$.

We can then proceed, as in method 1, to solve for A from the total area 145 of the blue stripes; from there we can solve for the total area of the red stripes.

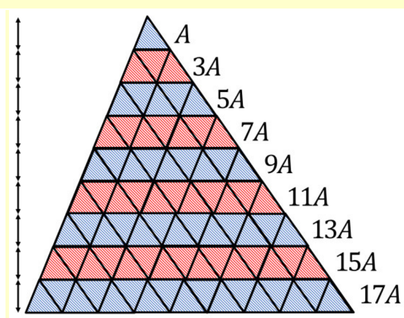


Figure 3

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