Triangle Projection Problem

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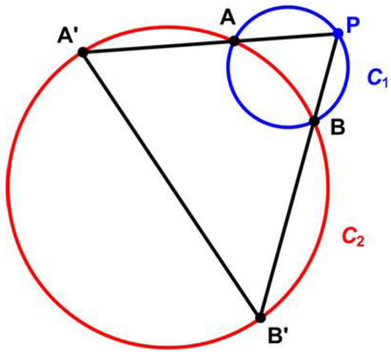

This is a Maths Item of the Month (MIOM) problem ([1]) that seems opaque at first. ("The Maths Item of the Month is a monthly problem aimed at teachers and students of GCSE and A level Mathematics.")

Two fixed circles, C_1 and C_2, intersect at A and B. P is on C_1. PA and PB produced meet C_2 at A' and B' respectively. How does the length of the chord $A'B'$ change as P moves?

Just start noticing relationships and the answer falls out nicely.

(MIOM problems often appear on MathsMonday and are also produced by Mathematics Education Innovation (MEI).)

Solution

Figure 1 shows added lines joining the labeled points and the angles between them. Since many of the angles subtend the same arcs, they are identical. Arranging the lines into various triangles we get a set of equations relating the angles.

$$\alpha + \gamma + \delta = 180° \quad (1)$$

$$\alpha + \gamma + \theta + \beta = 180°$$

Eliminating 180° between the equations yields

$$\delta = \theta + \beta$$

Similarly

$$\alpha + \delta + \psi + \beta = 180° \quad (2)$$

Eliminating 180° between the equations (1) and (2) yields

$$\gamma = \psi + \beta$$

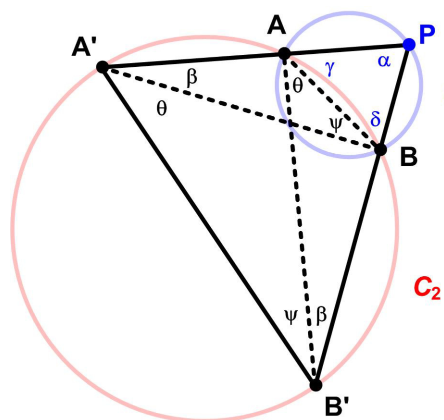

Figure 1

Therefore, triangle ABP is similar to triangle B'PA'. This means the length of segment A'B' is a constant scale factor times the length of segment AB. But since circles C_1 and C_2 don't move, the line AB doesn't move, which means its length is constant. Therefore the length of A'B' is also constant.

References

[1] "Circle Projection" *Maths Item of the Month*, July 2022. (https://mei.org.uk/maths-item-of-the-month/) No direct link provided anymore.