# Neuberg's Theorem

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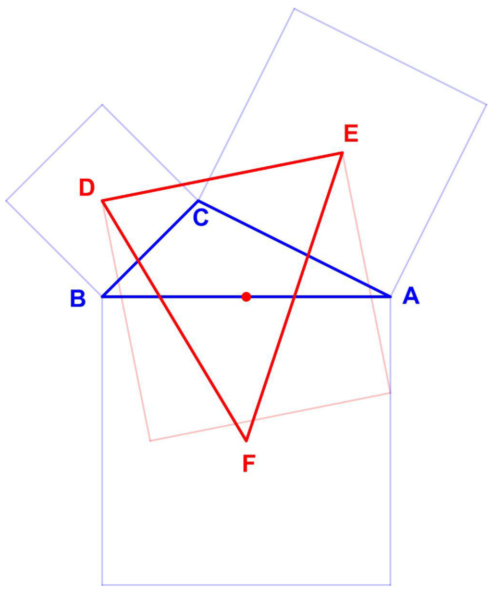


This turned out to be a challenging puzzle from the 1980 Canadian Math Society's magazine, *Crux Mathematicorum* ([1]).

*Proposed by Leon Bankoff, Los Angeles, California.*

Professor Euclide Paracelso Bombasto Umbugio has once again retired to his *tour d'ivoire* where he is now delving into the supersophisticated intricacies of the works of Grassmann, as elucidated by Forder's *Calculus of Extension.* His goal is to prove Neuberg's Theorem:

*If* D, E, F *are the centers of squares described externally on the sides of a triangle* ABC, *then the midpoints of these sides are the centers of squares described internally on the sides of triangle* DEF. [The accompanying diagram shows only one internally described square.]

Help the dedicated professor emerge from his self-imposed confinement and enjoy the thrill of hyperventilation by showing how to solve his problem using only highschool, synthetic, Euclidean, "plain" geometry.

Alas, my plane geometry capability was inadequate to solve the puzzle that way, so I had to resort to the sledge hammer of analytic geometry, trigonometry, and complex variables.

## My Solution

Figure 1 shows the case for the internally described square on one of the sides of the triangle DEF. The other sides are handled equivalently. It suffices to show the red triangle with vertex at the midpoint of side AB of the original triangle ABC and hypotenuse DE is an isosceles right triangle. We can then rotate the red triangle 3 times to fill the internally described square, showing that the center of the square is the midpoint of side AB.

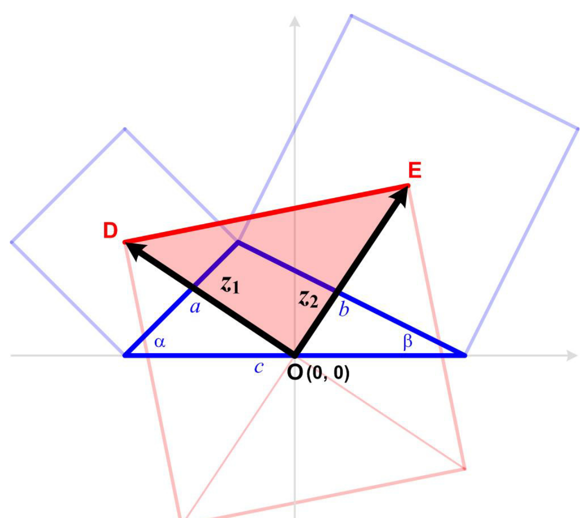


**Figure 1**

The angles and sides of the original triangle ABC are labeled as shown in Figure 1 with the origin of the coordinate system at the midpoint. The complex numbers $z_1$ and $z_2$ represent the vectors OD and OE. We wish to show they are perpendicular to each other and of the same length. (I was reminded of the complex variable approach by the similarities of this problem to Bottema's Theorem.)

My first approach involved an enormous amount of computation. I wanted to show perpendicularity via the Pythagorean Theorem, that is, $|z_1|^2 + |z_2|^2 = |z_2 - z_1|^2$. For complex numbers this involves complex conjugates:

$$|z_1|^2 + |z_2|^2 = z_1\overline{z_1} + z_2\overline{z_2}$$

$$|z_2 - z_1|^2 = (z_2 - z_1)(\overline{z_2} - \overline{z_1}) = z_1\overline{z_1} + z_2\overline{z_2} - (z_1\overline{z_2} + \overline{z_1}z_2) = z_1\overline{z_1} + z_2\overline{z_2} - 2\,\mathrm{Re}(z_1\overline{z_2})$$

So $|z_1|^2 + |z_2|^2 = |z_2 - z_1|^2$ if and only if $\mathrm{Re}(z_1\overline{z_2}) = 0$. To show $z_1$ and $z_2$ have the same length, I would have to show $z_1\overline{z_1} = z_2\overline{z_2}$. So three complicated products in all.

I managed to show $\mathrm{Re}(z_1\overline{z_2}) = 0$ after a lot of trial and error. But then I thought of a simpler way to get the answer in one equation without involving products, namely, show

$$z_1 = iz_2. \tag{1}$$

Recall that multiplication by $i = e^{i\frac{\pi}{2}}$ is just counter-clockwise rotation by $\pi/2 = 90°$.

From Figure 1 we have, since $e^{i\pi} = -1$,

$$z_1 = -\frac{c}{2} + \frac{\sqrt{2}}{2}ae^{i\left(\frac{\pi}{4}+\alpha\right)} \tag{2}$$

$$z_2 = \frac{c}{2} + \frac{\sqrt{2}}{2}be^{i\left(\pi-\left(\frac{\pi}{4}+\beta\right)\right)} = \frac{c}{2} - \frac{\sqrt{2}}{2}be^{-i\left(\frac{\pi}{4}+\beta\right)} \tag{3}$$

It certainly does not look like equations (2) and (3) satisfy equation (1). This is where we have to involve trigonometry. We are going to need some trigonometric relations from Figure 1, as well as some trigonometric properties:

$$c = a\cos\alpha + b\cos\beta \tag{4}$$

$$a\sin\alpha = b\sin\beta. \tag{5}$$

$$\cos(\tfrac{\pi}{4} + \theta) = \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta) \tag{6}$$

$$\sin(\tfrac{\pi}{4} + \theta) = \frac{\sqrt{2}}{2}(\cos\theta + \sin\theta) \tag{7}$$

$$\cos(\pi - \theta) = -\cos\theta \tag{8}$$

Now, using equations (4) – (8),

$$\mathrm{Re}(z_1) = -\frac{c}{2} + \frac{\sqrt{2}}{2}a\cos(\tfrac{\pi}{4} + \alpha) = \frac{1}{2}\left[-(a\cos\alpha + b\cos\beta) + (a\cos\alpha - a\sin\alpha)\right]$$

$$= -\frac{1}{2}\left[b\cos\beta + b\sin\beta\right]$$

$$\mathrm{Im}(z_1) = \frac{\sqrt{2}}{2}a\sin(\tfrac{\pi}{4} + \alpha) = \frac{1}{2}(a\cos\alpha + a\sin\alpha)$$

$$\mathrm{Re}(iz_2) = -\frac{\sqrt{2}}{2}b\sin(\tfrac{\pi}{4} + \beta) = -\frac{1}{2}(b\cos\beta + b\sin\beta)$$

$$\mathrm{Im}(iz_2) = \frac{c}{2} - \frac{\sqrt{2}}{2} b\cos(\tfrac{\pi}{4}+\beta) = \frac{1}{2}\left[(a\cos\alpha + b\cos\beta) - (b\cos\beta - b\sin\beta)\right]$$

$$= \frac{1}{2}\left[a\cos\alpha + a\sin\alpha\right]$$

and so equation (1), $z_1 = iz_2$, is true.

This was a nightmare of minus signs, which included such things as $(i)(-i) = -(-1) = 1$ and expressions that differed only by minus signs. It took a while to get it right.

## *Crux Mathematicorum* Solution

The solution is depressingly simple—again, *once you see it*. Shades of a coffin problem.

*Solution by J.T. Groenman, Arnhem, The Netherlands.*

It suffices to show that, say, the midpoint M of BC is the center of the square on EF or, equivalently, that ME = MF and ME ⊥ MF (see Figure 2) .

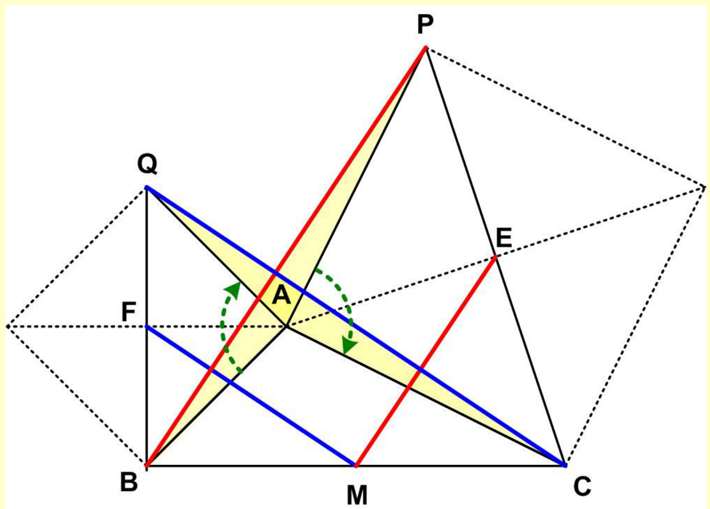


**Figure 2** [Augmented from the original]

A 90° rotation about A takes triangle PAB into triangle CAQ, so

PB = CQ and PB ⊥ CQ.

Now ME = MF follows from

ME = ½ BP = ½ CQ = MF

[via similar triangles scaled by 2] and ME ⊥ MF follows from

ME ∥ BP ⊥ CQ ∥ MF.

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W, DODGE, University of Maine at Orono; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; ANDY LIU, University of Alberta; LAI LANE LUEY, Willowdale, Ontario; LEROY F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; ROBERT TRANQUILLE, College de Maisonneuve, Montreal, Quebec; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

*Editor's comment.*

This theorem is credited to Neuberg by Forder,[1] who gives a two-line solution, using the method and notation of Grassmann, which so far has thoroughly defeated (and deflated) Professor Umbugio, the premier mathematician at the University of Guayazuela, despite frequent invocations to his patron saint and namesake Euclide.[2]

In an effort to help the good professor, most of our solvers submitted solutions (some quite lengthy) in "plain" geometry. Dodge and van de Craats gave proofs by transformation geometry and expressed the hope that such proofs would soon be considered "plain" geometry even by Professor Umbugio. In view of the parlous state of geometry in Guayazuela (and North America—we don't know about the rest of the world), that time is not yet. Garfunkel gave a proof by complex numbers, which could only be considered "plain" geometry in an imaginary high school.

There is only one statement in our featured solution which is likely to cause Professor Umbugio some concern: that PB ⊥ CQ results from a 90° rotation about A. Figure 3 is all that is needed to bring that statement into the realm of "plain" geometry.

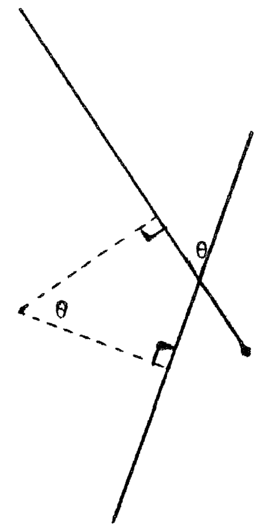

**Figure 3**

Ah, the genius of adding the lines PB and CQ.  They were the key.

# References


[1] Bankoff, Leon, "Problem 540," *Crux Mathematicorum*, Vol.6 No.4 April, Canadian Mathematical Society, 1980. p.114.

[2] Groenman, J. T., "Problem 540 Solution," *Crux Mathematicorum*, Vol.7 No.4 April, Canadian Mathematical Society, 1981. pp.127-8.


1 Henry George Forder, *The Calculus of Extension*, Chelsea, New York, 1960, p. 40.

2 JOS: More about the putative Professor Umbugio can be found at *Crux Mathematicorum*, Vol.3 No.5 May 1977, pp. 118-128. (https://cms.math.ca/publications/crux/issue?volume=3&issue=5)