Falling Sound Problem

Jim Stevenson

17 June 2022

This math problem from Colin Hughes's *Maths Challenge* website (mathschallenge.net) ([1]) hearkens back to basic physics.

A boy drops a stone down a well and hears the splash from the bottom after three seconds. Given that sound travels at a constant speed of 300 m/s and the acceleration of the stone due to gravity is 10 m/s^2, how deep is the well?

My Solution

If g = 10 m/s^2 is the constant acceleration of the stone falling from rest under the force of gravity, then the stone's velocity v after t seconds is, from integration,

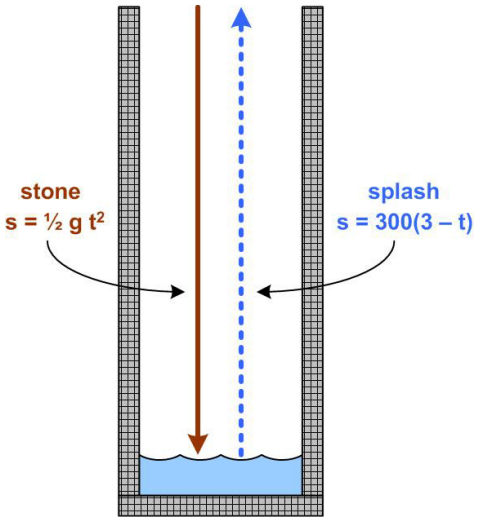

Figure 1

$$v = \int_0^t g\,dt = gt.$$

And the distance s the stone falls from rest, measuring positively downward, is, from another integration,

$$s = \int_0^t v\,dt = \int_0^t gt\,dt = \tfrac{1}{2} gt^2 = 5t^2.$$

We are given that the speed of sound is 300 m/s, so the distance traveled by the sound of the splash being the same as the distance the stone fell implies

$$s = 5t^2 = 300(3 - t)$$

where t is the time for the stone to hit the water at a depth of s and $3 - t$ is the remaining time for the sound of the splash to reach the top for a total drop to return-splash of 3 seconds. Therefore,

$$t^2 + 60\,t - 180 = 0$$

which has positive solution of

$$t = 6(\sqrt{30} - 5) \text{ sec.}$$

Therefore the depth of the well is

$$s = 300(3 - t) = 300(3 - 6(\sqrt{30} - 5)) = 300(33 - 6\sqrt{30}) = 40.99 = 41 \text{ m}$$

Maths Challenge Solution

This solution is essentially the same as mine.

From the moment the stone is dropped to the splash heard three seconds later two distinct events occur: the stone takes t seconds to hit the water below and the sound takes $3 - t$ seconds to travel back up the well.

If s is the depth of the well then we can use $s = ut + at^2/2$, where u is initial velocity and a is acceleration. During the stone's descent, $s = 5t^2$, as the initial velocity is zero and we can ignore the direction of acceleration as we are only concerned with distance rather than displacement. As the sound of the splash travels back up the well, $s = 300(3 - t)$ (acceleration of sound is zero).

Therefore $5t^2 = 300(3 - t)$, leading to the quadratic, $t^2 + 60t - 180 = 0$. Solving this we take the positive solution, $t = 6\sqrt{30} - 30$ seconds. Using $s = 300(3 - t)$ we get the depth of the well, $s = 300(33 - 6\sqrt{30}) \approx 41.0$ metres.

References

[1] Hughes, Colin, "Falling Sound", *Maths Challenge*, (mathschallenge.net) (21 September 2008) #347 p.11. Difficulty: 3 Star. "A three-star problem: a good knowledge of school mathematics and/or some aspects of proof will be required."