Broken Diagonal Problem

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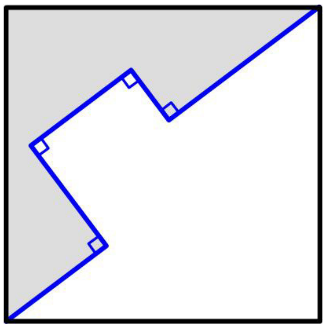

This is a nice problem from the UKMT Senior Mathematics Challenge for 2022 ([1]):

Five line segments of length 2, 2, 2, 1 and 3 connect two corners of a square as shown in the diagram. What is the shaded area?

A 8 B 9 C 10 D 11 E 12

The pleasure of solving this problem may be lessened if one is under a time crunch, as is the case with all these timed tests.

My Solution

I decided to superimpose a 1x1 grid on the figure to help reveal the relationships (Figure 1). I then added a red line to divide the square into halves. The desired shaded region and a yellow region, that can be seen to be 3½ square units, make up one half of the square.

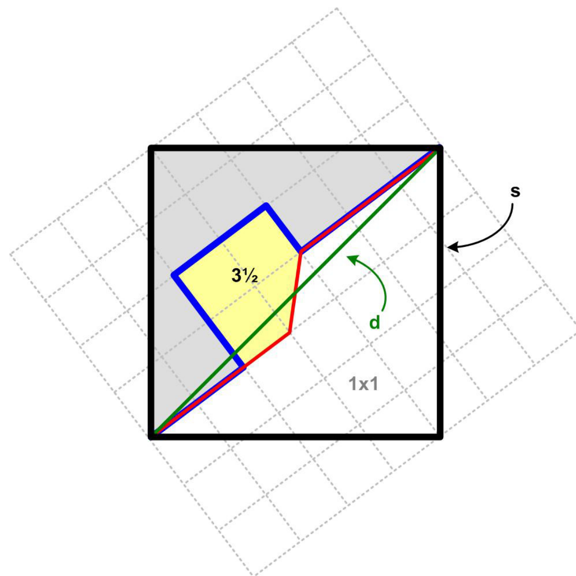

Figure 1

A green diagonal line is drawn to again divide the square into two halves. We see from the figure and the Pythagorean Theorem that

$$2s^2 = d^2 = 1^2 + 7^2 = 50.$$

So $$s^2 = 25.$$

Then the shaded area is

$$\tfrac{1}{2} s^2 - 3\tfrac{1}{2} = 12\tfrac{1}{2} - 3\tfrac{1}{2} = 9 \text{ (Answer B)}$$

UKMT Solution

Answer B

By identifying similar right-angled triangles, we can first calculate the side-length of the large square. Drawing an extra line *RU* to complete rectangle *RTSU* gives *SR* = 1 and *RV* = 5. A straight line from *O* to *V* passes through *SP* at *Q*. Let *PQ* = *x* and therefore *QR* = 1 − *x*. As ∠*OQP* and ∠*VQR* are vertically opposite, they are equal, so triangle *OQP* and triangle *VQR* are similar. Therefore *x*/2 = (1 − *x*)/5 which rearranges to give *x* = 2/7. The ratio *PQ* : *QR* = 2 : 5 and so the ratio *OQ* : *QV* = 2 : 5. This gives *OV* = 7/2 × *OQ*. Using Pythagoras' Theorem,

$$OV = \frac{7}{2}\sqrt{2^2 + \left(\frac{2}{7}\right)^2} = 5\sqrt{2}.$$

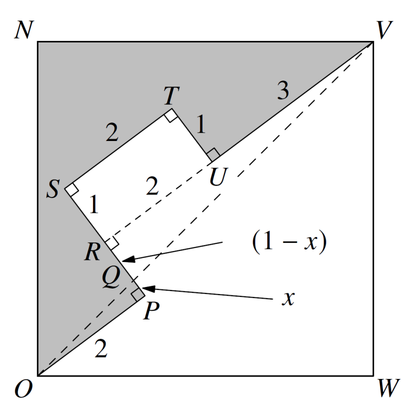

Figure 2

So *OW* = *VW* = 5. The shaded area is then area of triangle VNO − area of rectangle $RSTU$ − area of triangle VRQ + area of triangle OPQ

$= (\frac{1}{2} \times 5 \times 5) - (1 \times 2) - (\frac{1}{2} \times (2 + 3) \times {}^{5}/_{7}) + (\frac{1}{2} \times 2 \times {}^{2}/_{7})$

$= 9.$

References

[1] "Senior Mathematics Challenge Problem 25", United Kingdom Mathematics Trust, 4 October 2022. (https://www.ukmt.org.uk/sites/default/files/ukmt/SMC%202022%20Paper.pdf)

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