More Squares in Semicircle

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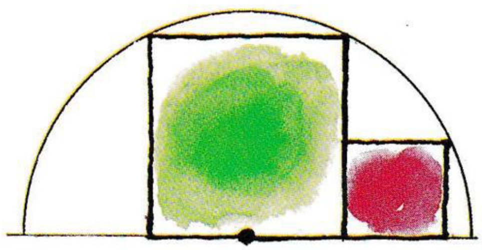

Here is another elegant *Quantum* math magazine Brainteaser problem ([1] p.19).

Two squares are inscribed in a semicircle as shown in the figure at left. Prove that the area of the big square is four times that of the small one.

My Solution

First, we need to show the midpoint of the bottom edge of the large square is the center of semicircle (Figure 1). The perpendicular bisector of the top edge of the square goes through center of circle. The bisector is parallel to the sides of square, and so implies it bisects the bottom edge of square on the diameter. Therefore the midpoint of the bottom edge is the center of circle.

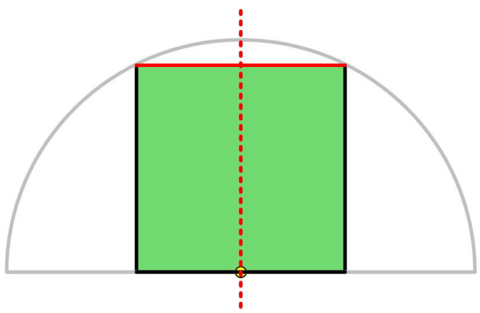

Figure 1

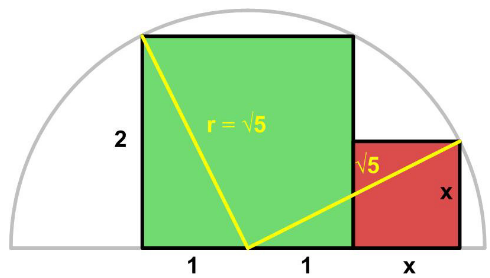

Figure 2

Next we draw radii of the semicircle to the top corners of each square as shown in Figure 2. We make the edge of the large square be 2 units and the unknown edge of the small square *x* units. Then the length of the radius is √5, which implies by the Pythagorean theorem

$$(1 + x)^2 + x^2 = 5 \quad\Rightarrow\quad x^2 + x - 2 = 0 \quad\Rightarrow\quad (x - 1)(x + 2) = 0 \quad\Rightarrow\quad x = 1$$

Therefore the ratio of the area of the large square to the small square is $4 : x^2$ or 4 : 1, as required.

Quantum Solution

The Quantum solution ([1] p.58) is slick.

It's obvious that in Figure 3, obtained by 90° rotation of the given big square about the circle's center, OC = OA = AB/2 = CD/2. So AF = AE (= AB/2), and AEDF is the given small square, whose side lengths are 1/2 those of the big one.

For the solution to be correct, one still needs to prove the midpoint of the bottom edge of the large square is the center of the semicircle, so that when the square is rotated, the edges intersect at the midpoints.

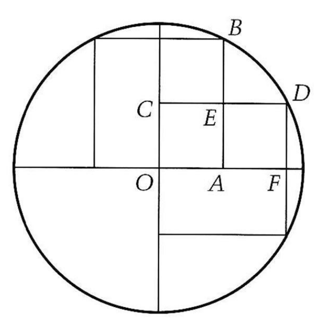

Figure 3 Quantum Solution

References

[1] “Squares in Semicircle” B94 “Brainteasers” *Quantum* Vol.4, No.1, National Science Teachers Assoc., Springer-Verlag, Sep-Oct 1993. p.19