Skating Rendezvous Problem

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This is a fun problem from the 1989 American Invitational Mathematics Exam (AIME) ([1]).

Two skaters, Allie and Billie, are at points *A* and *B*, respectively, on a flat, frozen lake. The distance between *A* and *B* is 100 meters. Allie leaves *A* and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with *AB*. At the same time Allie leaves *A*, Billie leaves *B* at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?

My Solution

Since Allie and Billie leave at the same moment and meet per force at the same time t, the vertical component of their velocity vectors must be the same (Figure 1).[1] Since Allie travels at a 60° angle with respect to their baseline, the components of her velocity vector must be

$$v^A{}_V = 8 \sin 60° = 8\sqrt{3}/2 = 4\sqrt{3} \text{ m/s}$$

$$v^A{}_H = 8 \cos 60° = 8/2 = 4 \text{ m/s}$$

Therefore the components of Billie's velocity vector must be

$$v^B{}_V = 4\sqrt{3} \text{ m/s}$$

$$v^B{}_H = \sqrt{(7^2 - (4\sqrt{3})^2)} = 1 \text{ m/s}$$

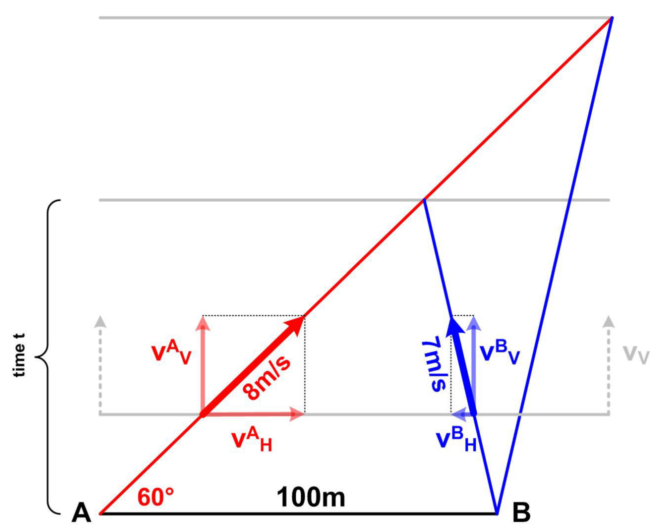

Figure 1

Since we are seeking the shortest time t when Allie and Billie meet, Billie must be skating toward Allie rather than away. Therefore,

$$v^A{}_H\, t = 100 - v^B{}_H\, t \;\Rightarrow\; t = 100/5 = 20 \text{ sec}$$

And so the distance Allie must skate before meeting Billie is 8·20 = 160 meters.

AIME Solutions

It is perhaps not surprising that there are a number of alternative solutions possible.

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[1] See the "Four Travelers Problem" (http://josmfs.net/2019/01/01/the-four-travelers-problem/) for a similar idea.

Solution 1

Label the point of intersection as C [Figure 2]. Since $d = rt$, $AC = 8t$ and $BC = 7t$. According to the law of cosines,

$$(7t)^2 = (8t)^2 + 100^2 - 2 \cdot 8t \cdot 100 \cdot \cos 60°$$

$$0 = 15t^2 - 800t + 10000 = 3t^2 - 160t + 2000$$

$$t = (160 \pm \sqrt{(160^2 - 4 \cdot 3 \cdot 2000)})/6 = 20,\ 100/3$$

Since we are looking for the earliest possible intersection, 20 seconds are needed. Thus, $8 \cdot 20 = 160$ meters is the solution.

Alternatively, we can drop an altitude from C and arrive at the same answer.

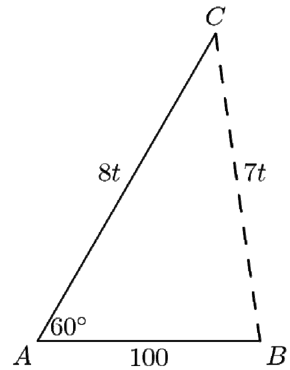

Figure 2

Solution 2

Let P be the point of intersection between the skaters, Allie and Billie. We can draw a line that goes through P and is parallel to AB. Letting this line be the x-axis, we can reflect B over the x-axis to get B'. As reflections preserve length, $B'X = XB$ [Figure 3].

We then draw lines BB' and PB'. We can let the foot of the perpendicular from P to BB' be X, and we can let the foot of the perpendicular from P to AB be Y. In doing so, we have constructed rectangle $PXBY$.

By $d = rt$, we have $AP = 8t$ and $PB = 7t$, where t is the number of seconds it takes the skaters to meet. Furthermore, we have a 30-60-90 triangle PAY, so $AY = 4t$, and $PY = 4t\sqrt{3}$. Since we have $PY = XB = B'X$, $B'X = 4t\sqrt{3}$. By Pythagoras, $PX = t$.

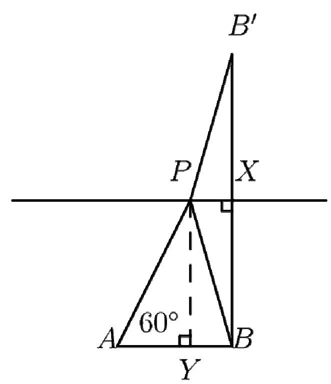

Figure 3

As $PXBY$ is a rectangle, $PX = YB$. Thus $AY + YB = AB \Rightarrow AY + PX = AB$, so we get $4t + t = 100$. Solving for t, we find $t = 20$.

Our answer, AP, is equivalent to $8t$. Thus, $AP = 8 \cdot 20 = 160$.

Solution 3

We can define x to be the time elapsed since both Allie and Billie moved away from points A and B respectfully. Also, set the point of intersection to be M.

Then we can produce the following diagram:

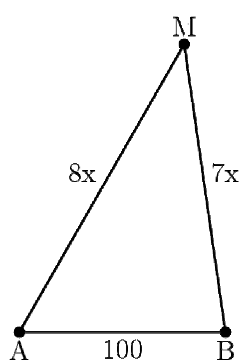

Figure 4

Now, if we drop an altitude from point M, we get :

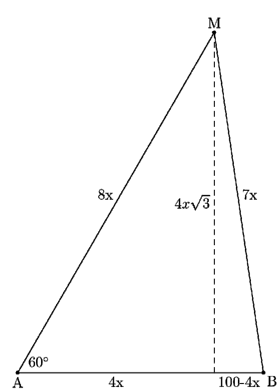

Figure 5

We know this from the 30-60-90 triangle that is formed. From this we get that:

$$(7x)^2 = (4\sqrt{3})^2 + (100 - 4x)^2 \quad \Rightarrow \quad 49x^2 - 48x^2 = x^2 = (100 - 4x)^2$$

$$\Rightarrow\ 0 = (100 - 4x)^2 - x^2 = (100 - 3x)\ (100 - 5x)$$

Therefore, we get that $x = 100/3$ or $x = 20$. Since $20 < 100/3$, we have that $x = 20$ (since the problem asks for the quickest possible meeting point), so the distance Allie travels before meeting Billie would be $8x = 8 \cdot 20 = 160$ meters.

References

[1] "Problem 6" 1989 AIME Problems
(https://artofproblemsolving.com/wiki/index.php/1989_AIME_Problems)