A Nice Factorial Sum

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This is another infinite series from Presh Talwalkar,[1] but with a twist.

This problem is adapted from one given in an annual national math competition exam in France. Evaluate the infinite series:

1/2! + 2/3! + 3/4! + …

The twist is that Talwalkar provides three solutions, illustrating three different techniques that I in fact have used before in series and sequence problems. But this time I actually found a simpler solution that avoids all these. You also need to remember what a factorial is: $n! = n(n-1)(n-2)\cdot \ldots \cdot 3\cdot 2\cdot 1$.

My Solution

We can write the series as follows (we assume all series converge):

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}+\ldots=\sum_{n=2}^{\infty}\frac{n-1}{n!}$$

$$=\sum_{n=2}^{\infty}\frac{n}{n!}-\sum_{n=2}^{\infty}\frac{1}{n!}$$

$$=\sum_{n=2}^{\infty}\frac{1}{(n-1)!}-\sum_{n=2}^{\infty}\frac{1}{n!}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n!}-\sum_{n=2}^{\infty}\frac{1}{n!}$$

$$=1+\sum_{n=2}^{\infty}\frac{1}{n!}-\sum_{n=2}^{\infty}\frac{1}{n!}$$

$$=1$$

Simple! (Actually, I guess my solution is essentially Talwalkar's "Method 2: telescoping sum", but in perhaps a simpler guise.)

Talwalkar's Solutions

When I saw the problem, I worked it out by mathematical induction. Then I researched methods and found a post on Quora[2] with solutions using a telescoping sum and a power series. I will present all three ways since it is always good to know different problem solving methods.

[1] 4 Oct 2021 https://mindyourdecisions.com/blog/2021/10/04/a-nice-factorial-sum-from-france/

[2] https://www.quora.com/What-is-sum-of-the-series-1-2-+2-3-+3-4-+-to-infinity

Method 1: mathematical induction

Consider the partial sum:

$$S(n) = 1/2! + 2/3! + 3/4! + \ldots + n/(n+1)!$$

We can calculate some values to establish a pattern:

$$S(1) = 1/2! = 1/2 = 1 - 1/2!$$

$$S(2) = S(1) + 2/3! = 1/2 + 2/6 = 5/6 = 1 - 1/3!$$

$$S(3) = S(2) + 3/4! = 5/6 + 3/24 = 23/24 = 1 - 1/4!$$

From the calculations, we might conjecture a formula for the partial sum is:

$$S(n) = 1 - n/(n+1)!$$

We will prove this by induction. We have established the base cases of $n = 1, 2, 3$. We now suppose the formula is true for some $n = k$, and we will show that if $S(k)$ is true then $S(k + 1)$ is true.

$$S(k+1) = S(k) + (k+1)/(k+2)!$$

By the induction hypothesis we have $S(k) = 1 - 1/(k+1)!$, so we can simplify:

$$S(k) + (k+1)/(k+2)! = 1 - 1/(k+1)! + (k+1)/(k+2)!$$

Now we will multiply the second term by $(k + 2)/(k + 2)$ and simplify:

$$1 - (k+2)/[(k+1)!(k+2)] + (k+1)/(k+2)! = 1 - (k+2)/(k+2)! + (k+1)/(k+2)!$$

$$= 1 - 1/(k+2)!$$

This verifies the second step of induction, and thus the induction formula is true.

We then have the infinite series S is the limit of $S(n)$ as n goes to infinity. Since $1/(n + 2)!$ will go to 0 as n goes to infinity, we will have:

$$S = 1 - 0 = 1$$

Thus the infinite series has a value equal to 1.

Method 2: telescoping sum

We again start by considering the partial sum.

$$S(n) = 1/2! + 2/3! + 3/4! + \ldots + n/(n+1)!$$

We now will re-write each term in the sum using a trick of adding and subtracting 1 to the numerator.

$$\begin{aligned} k/(k+1)! &= (k+1-1)/(k+1)! \\ &= (k+1)/(k+1)! - 1/(k+1)! \\ &= 1/k! - 1/(k+1)! \end{aligned}$$

We can apply this formula to each term in the sum to get:

$$\begin{aligned} S(n) &= 1/2! + 2/3! + 3/4! + \ldots + n/(n+1)! \\ &= 1/1! - 1/2! + 1/2! - 1/3! + 1/3! - 1/4! + \ldots + 1/n! - 1/(n+1)! \\ &= 1/1! + (-1/2! + 1/2!) + (-1/3! + 1/3!) + (-1/4! + 1/4!) + \ldots + (-1/n! + 1/n!) - 1/(n+1)! \\ &= 1/1! - 1/(n+1)! \end{aligned}$$

$= 1 - 1/(n + 1)!$

We thus have the same partial sum as method 1. We then take the limit as n goes to infinity to conclude the sum of the infinite series is equal to 1.

Method 3: power series

Since the sum involves denominators with increasing values of the factorial function, we can think about the power series for the exponential function.

$$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + \ldots$$

We somehow want to get the infinite series with a general term $k/(k + 1)!$. To do that, we will divide both sides by x and then take the derivative of both sides.

$$e^x/x = 1/x + 1/1! + x/2! + x^2/3! + x^3/4! + \ldots$$

Now we take the derivative of each side to get:

$$(xe^x - e^x)/x^2 = -1/x^2 + 0 + 1/2! + 2x/3! + 3x^2/4! + \ldots$$

We then substitute $x = 1$.

$$(e^1 - e^1)/1^2 = -1/1 + 0 + 1/2! + 2/3! + 3x^2/4! + \ldots$$

$$0 = -1 + 1/2! + 2/3! + 3/4! + \ldots$$

$$0 = -1 + S$$

$$S = 1$$

Thus the infinite series has a value equal to 1–what an incredible way to solve this problem!

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Reference

I learned about methods 2 and 3 on Quora

https://www.quora.com/What-is-sum-of-the-series-1-2-+2-3-+3-4-+-to-infinity

[3] http://www.patreon.com/mindyourdecisions