Parallel Stroll Problem

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This is a slightly challenging problem from the 1993 American Invitational Mathematics Exam (AIME) ([1]).

Jenny and Kenny are walking in the same direction, Kenny at 3 feet per second and Jenny at 1 foot per second, on parallel paths that are 200 feet apart. A tall circular building 100 feet in diameter is centered midway between the paths. At the instant when the building first blocks the line of sight between Jenny and Kenny, they are 200 feet apart. Find *t*, the amount of time in seconds, before Jenny and Kenny can see each other again.

My Solution

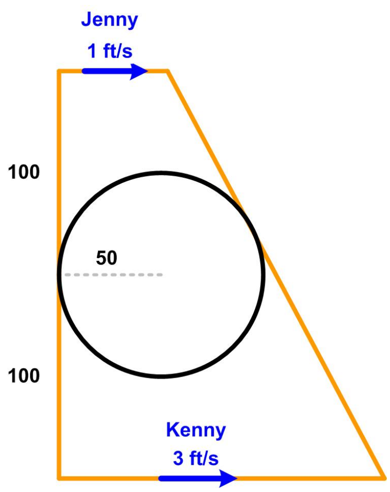

Figure 1

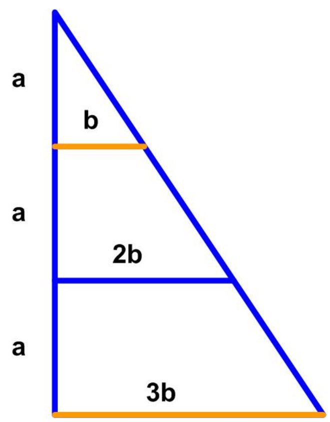

Figure 2

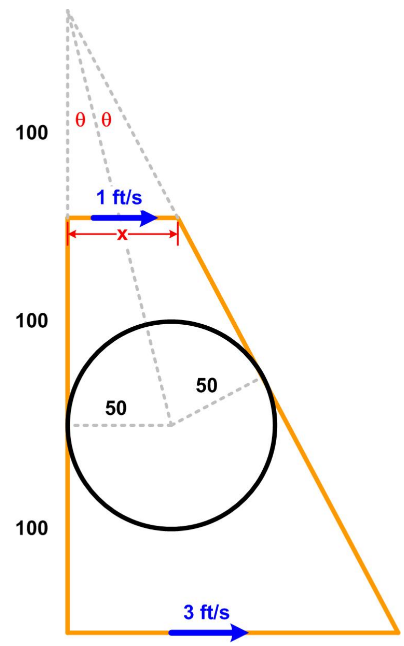

Figure 3

Figure 1 shows the problem setup. Jenny and Kenny are initially opposite each other at 200 feet just as they begin passing the circular building. At time *t* they have just seen each other on the other side of the building. This means their line of sight is tangent to the circular building. Furthermore, Kenny has advanced 3 times the distance that Jenny has.

Figure 2 indicates that if we extend the slanted tangent until it intersects the vertical tangent to the circle, we get a triangle with the properties that evenly-spaced lines will be in the ratios of 1 : 2 : 3 (from the corresponding similar triangles). Therefore, we imbed the original figure in a right triangle as shown in Figure 3. The added dashed lines show we have two congruent right triangles, since tangents are perpendicular to radii and two equal sides imply the third sides are equal. Therefore, we have the two equal angles θ as shown. From this we can compute the distance *x* Jenny travels up to the moment *t* when she sees Kenny, namely, $x = 100 \tan 2\theta$. But $\tan \theta = 50/200 = ¼$. So

$$x = 100 \tan 2\theta = 100 \cdot \frac{2\tan\theta}{1-\tan^2\theta} = 100 \cdot \frac{2\cdot\frac{1}{4}}{1-(\frac{1}{4})^2} = 100 \cdot \frac{8}{15} = \frac{160}{3} = 53\tfrac{1}{3} \text{ feet}$$

Since Jenny is traveling at 1 ft/s, the time when they see each other again is 53 ⅓ seconds.

AIME Solutions

The solutions had assumed the units were meters instead of feet. So I changed the meters back to feet. Given the idiosyncrasy of the AIME, fractions were reduced to integers by summing the numerator and denominator. I left the answers in fractions.

Solution 1

Consider the unit circle of radius 50.[1] Assume that they start at points (–50, 100) and (–50, –100). Then at time t, they end up at points (–50 + t, 100) and (–50 + 3t, –100). The equation of the line connecting these points and the equation of the circle are

$$y = -100x/t + 200 - 5000/t \qquad (1)$$

$$50^2 = x^2 + y^2 \qquad (2)$$

When they see each other again, the line connecting the two points will be tangent to the circle at the point (x, y). Since the radius is perpendicular to the tangent we get $-x/y = -100/t$ or $xt = 100y$. Now substitute $y = xt/100$ into (2) and get $x = 5000/\sqrt{(100^2 + t^2)}$. Now substitute this and $y = xt/100$ into (1) and solve for t to get $t = 160/3$.

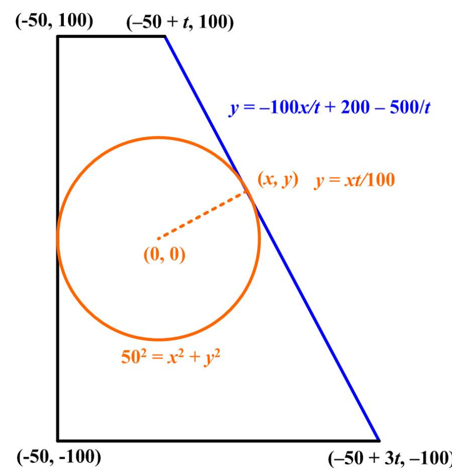

Figure 4 (added by JOS)

Solution 2

Essentially my solution.

Let A and B be Kenny's initial and final points respectively and define C and D similarly for Jenny. Let O be the center of the building. Also, let X be the intersection of AC and BD. Finaly, let P and Q be the points of tangency of circle O to AC and BD respectively.

From the problem statement, $AB = 3t$ and $CD = t$. Since $\Delta ABX \sim \Delta CDX$,

$$CX = AC \cdot \left(\frac{CD}{AB - CD}\right) = 200 \cdot \left(\frac{t}{3t - t}\right) = 100.$$

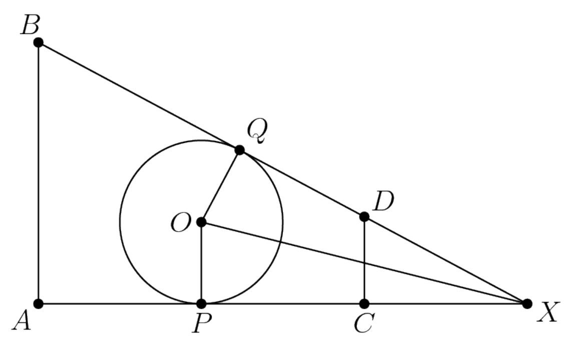

Figure 5

Since $PC = 100$, $PX = 200$. So, $\tan(\angle OXP) = OP/PX = 50/200 = ¼$. Since circle O is tangent to BX and AX, OX is the angle bisector of $\angle BXA$. Thus,

$$\tan(\angle BXA) = \tan(2\angle OXP) = \frac{\tan(\angle OXP)}{1-\tan^2(\angle OXP)} = \frac{2\cdot\frac{1}{4}}{1-(\frac{1}{4})^2} = \frac{8}{15}.$$

Therefore, $t = CD = CX \cdot \tan(\angle BXA) = 100 \cdot 8/15 = 160/3$.

[1] JOS: Not a *unit* circle then! AIME means a circle of radius 50 *centered at the origin*.

Solution 3

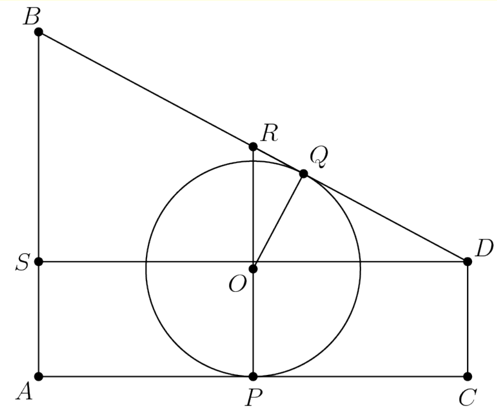

Figure 6

Let t be the time they walk. Then $CD = t$ and $AB = 3t$. Draw a line from point O to Q such that OQ is perpendicular to BD. Further, draw a line passing through points O and P, so OP is parallel to AB and CD and is midway between those two lines. Then

$$PR = (AB + CD)/2 = (3t + t)/2 = 2t.$$

Draw another line passing through point D and parallel to AC, and call the point of intersection of this line with AB as S. Then

$$SB = AB - AS = 3t - t = 2t.$$

We see that $m\angle SBD = m\angle ORQ$ since they are corresponding angles, and thus by angle-angle similarity, $\Delta QOR \sim \Delta SDB$. Then

$$OQ/DS = RO/BD \Rightarrow 50/200 = RO / \sqrt{(200^2 + 4t^2)}$$
$$\Rightarrow RO = \sqrt{(200^2 + 4t^2)} / 4$$
$$\Rightarrow RO = \sqrt{(100^2 + t^2)} / 2$$

And we obtain

$$PR - OP = RO$$
$$2t - 50 = \sqrt{(100^2 + t^2)} / 2$$
$$4t - 100 = \sqrt{(100^2 + t^2)}$$
$$(4t - 100)^2 = (\sqrt{(100^2 + t^2)})^2$$
$$16t^2 - 800t + 100^2 = t^2 + 100^2$$
$$15t^2 = 800t$$
$$t = 800/15$$

so we have $t = 160/3$, and our answer is $160 + 3 = 163$.

Solution 4

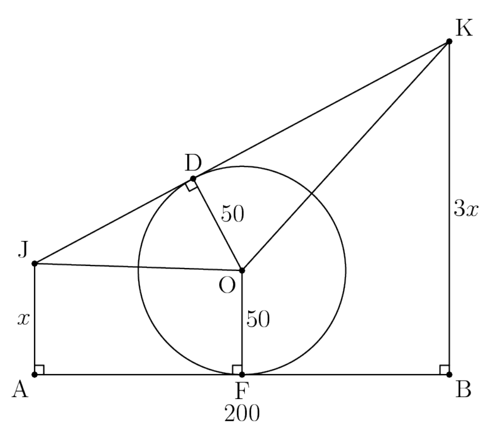

Figure 7

We can use areas to find the answer. Since Jenny and Kenny are 200 feet apart, we know that they are side by side, and that the line connecting the two of them is tangent to the circular building. We know that the radius of the circle is 50, and that $AJ = x$, $BK = 3x$.

By areas,

$$[OJK] + [AJOF] + [OFBK] = [ABKJ].$$

Having right trapezoids, $[AJOF] = 100\cdot(x + 50)/2$. The other areas of right trapezoids can be calculated in the same way. We just need to find $[OJK]$ in terms of x.

If we bring AB up to where the point J is, we have by the Pythagorean Theorem,

$$JK = 2\sqrt{(x^2 + 10000)} \;\Rightarrow\; [OJK] = JK \cdot OD / 2.$$

Now we have everything to solve for x.

$$[OJK] + [AJOF] + [OFBK] = [ABKJ]$$

$$50\sqrt{x^2 + 10000} + \frac{x+50}{2}\cdot 100 + \frac{3x+50}{2}\cdot 100 = \frac{x+3x}{2}\cdot 200$$

After isolating the radical, dividing by 50, and squaring, we obtain:

$$15x^2 - 800x = 0 \;\Rightarrow\; x = 160/3.$$

Since Jenny walks 160/3 feet at 1 ft/s, our answer is 163/3 seconds.

Solution 5

Basically, draw out a good diagram, and the rest is done.

References

[1] "Problem 13" 1993 AIME Problems
(https://artofproblemsolving.com/wiki/index.php/1993_AIME_Problems)

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