An Intercept Problem

27 October 2021

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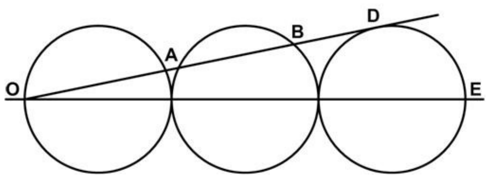


This is a straight-forward problem by Geoffrey Mott-Smith from 1954 ([1]).

Three tangent circles of equal radius r are drawn, all centers being on the line OE. From O, the outer intersection of this axis with the left-hand circle, line OD is drawn tangent to the right-hand circle. What is the length, in terms of r, of AB, the segment of this tangent which forms a chord in the middle circle?

My Solution

Figure 1 represents a parametrization of the problem based on the information given. The perpendicular distance from the center of the middle circle to the line OD is given by x. Note that x lies on the perpendicular *bisector* of the segment AB, since that line passes through the center of the circle.

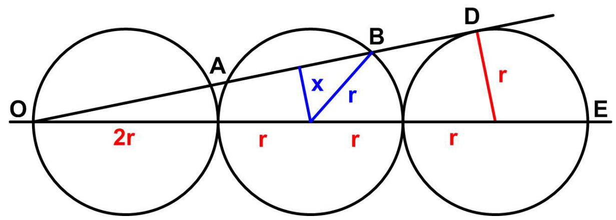


Figure 1

Then we have the following relations.

$$x / 3r = r / 5r = 1/5 \Rightarrow x = 3r/5$$

So,

$$(AB/2)^2 + (3r/5)^2 = r^2,$$

which implies

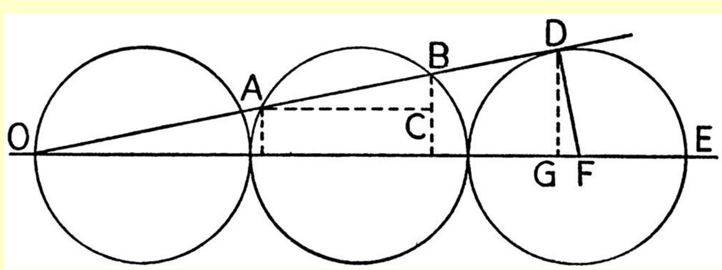
$$AB = 8r/5.$$

Mott-Smith Solution

His solution is a bit more convoluted.

The length AB is equal to $8r/5$.

The computation involves analytical algebra. Determine the equation of the tangent OD and the equation of the middle circle; solve these equations simultaneously to find the points of intersection A, B; then compute the distance between these points.



Take O as the center of rectangular co-ordinates, with OE as the x-axis. The equation of line OD is then given by the ratio of DG (perpendicular from D to OE) to OG. Since ODF is a right triangle, the altitude DG is equal to (OD)(DF) / OF. DF equals r and OF equals $5r$; from these values the other terms can be computed. The equation of OD is found to be

$$y / x = \sqrt{6} / 12$$

The equation of the middle circle is

$$(x - 3r)^2 + y^2 = r^2.$$

Solve the two equations simultaneously for the value of x, which is found to be

$$x = 72r / 25 \pm (8r / 25) \sqrt{6}$$

The two values of x are the abscissas of the points B and A. The difference between these values, equal to $(16r / 25) \sqrt{6}$, is the length of AC. Since ABC is a right triangle by construction, AB can be computed from AC and BC.[1] The latter can be computed by use of the equation for OD. The desired length AB is found to be $8r/5$.

Comment 1. For once I seemed to have found a shorter solution than the proposer. Of course, it depends on the knowledge that the perpendicular bisector of a line segment between two points on a circle passes through the center of the circle. But this is a basic fact, easily proved.

To wit, consider a (green) line segment bisected by a perpendicular (blue) line that does *not* pass through the center of the circle with (red) radius r (Figure 2).

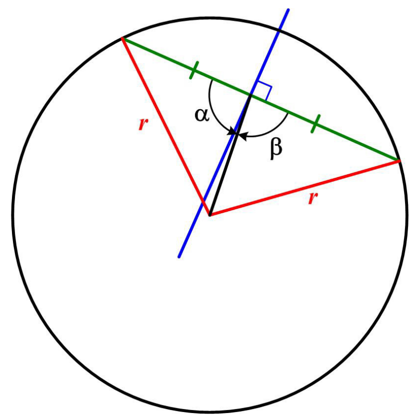


Figure 2

Draw a (black) line segment from the center of the circle to the bisection point on the green line. This line, together with the green halves of the line segment and the red radial lines, forms two triangles, which have their sides equal and therefore are congruent. That means angle α equals angle β. Since these angles sum to form a straight line, we must have $\alpha + \beta = 180°$, or $\alpha = \beta = 90°$. Therefore, the black line is also perpendicular to the green line and passes through the bisection point of the green line. So it must coincide with the blue line, which implies that the assumption that the perpendicular bisector of the green line does not pass through the center of the circle is false.

Comment 2. Actually, I guess I used the converse. That is, I didn't start with a perpendicular bisector and needed it to go through the center, but rather started with a perpendicular through the center and needed it to bisect the line. But this is even easier. The two right triangles with the radii as hypotenuses and a common leg immediately imply, via Pythagoras, that the third sides are equal, and so the line is bisected.

References

[1] Mott-Smith, Geoffrey, "101. An Intercept Problem," *Mathematical Puzzles for Beginners & Enthusiasts*, Blakiston Co, 1946, 2nd revised edition, Dover Publications, 1954.

[1] JOS: BC equals the y-coordinate at B minus the y-coordinate at A, or, via $y = x\sqrt{6}/12$,

$$BC = (\sqrt{6}/12)(72r/25 + (8r/25)\sqrt{6}) - (\sqrt{6}/12)(72r/25 - (8r/25)\sqrt{6}) = (\sqrt{6}/12)(16r/25)\sqrt{6} = 8r/25$$