Winter Sum

13 September 2021

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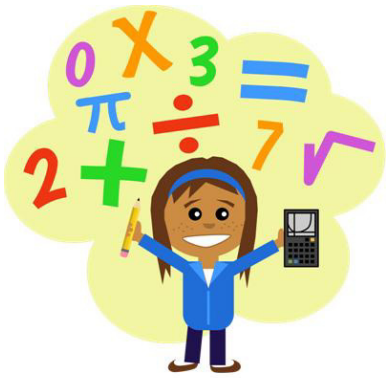

Here is another sum problem, this time from the 2021 Math Calendar ([1]).

$$\frac{7}{3}+\frac{9}{9}+\frac{11}{27}+\frac{13}{81}+\ldots$$

As before, recall that all the answers are integer days of the month. And the solution employs a technique familiar to these pages.

Solution

Let S be the sum. Then

$$S=\frac{7}{3^1}+\frac{9}{3^2}+\frac{11}{3^3}+\frac{13}{3^4}+\ldots+\frac{2n+5}{3^n}+\ldots$$

that is,

$$S=\sum_{n=1}^{\infty}\frac{2n+5}{3^n}=2\sum_{n=1}^{\infty}\frac{n}{3^n}+5\sum_{n=1}^{\infty}\frac{1}{3^n}$$

so

$$S(x)=2\sum_{n=1}^{\infty}nx^n+5\sum_{n=1}^{\infty}x^n$$

where $S = S(1/3)$.

Again we use the power series approach with the geometric series:

$$G(x)=1+x+x^2+\ldots+x^n+\ldots=\frac{1}{1-x}$$

Taking the derivative,

$$G'(x)=1+2x+3x^2+\ldots+nx^{n-1}+\ldots=\frac{1}{(1-x)^2}$$

Then

$$S(x)=2xG'(x)+5(G(x)-1)=\frac{2x}{(1-x)^2}+\frac{5x}{1-x}$$

Evaluating at $x = 1/3$,

$$\boxed{S=\frac{3}{2}+\frac{5}{3}\cdot\frac{3}{2}=4}$$

Comment

I have used this trick of converting an infinite series into a power series involving operations on the geometric series so often that I am reminded of a quip whose origin I have long forgotten: "A trick that is used over and over becomes a technique." This sort of echoes a quote by Rene Descartes (1621) in *Fermat's Library*:[1]

"Each problem that I solve becomes a rule which serves afterwards to solve other problems."

Of course, I am not alone using this approach. In his final Lockdown Math lecture Grant Sanderson at *3Blue1Brown* includes this technique in a more advanced setting in his excellent discussion of problem solving techniques ([2]). Sanderson presents in considerable detail the solution to two problems by way of illustrating some principles to problem solving that are reminiscent of those given by George Polya in his famous book, *How To Solve It* (1945). I have often referred to Polya in solutions on this website. His book is an indispensable addition to the shelf of every mathematician and math student.

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2021: Your Daily epsilon of Math*, Rock Point, Quarto Publishing Group, New York, 2021. February

[2] Sanderson, Grant, "Tips to be a better problem solver [Last live lecture] | Ep. 10 Lockdown live math", *3Blue1Brown*, 22 May 2020 (https://www.youtube.com/watch?v=QvuQH4_05LI)

[1] https://twitter.com/fermatslibrary/status/1478006243113508866, 9:12 AM · Jan 3, 2022