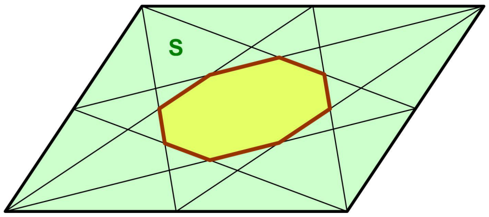
Octagonal Area Problem

11 August 2021

Jim Stevenson



Here is another problem from the Polish Mathematical Olympiads published in 1960 ([1]).

95. In a parallelogram of given area *S* each vertex has been connected with the mid-points of the opposite two sides. In this manner the parallelogram has been cut into parts, one of them being an octagon. Find the area of that octagon.

My Solution

First, label the altitude *h* and base *w* of the parallelogram as in Figure 1. Then the area of the parallelogram is $S = hw$. Now apply a horizontal shear to the parallelogram to obtain a rectangle with the same altitude and base (Figure 2). If you imagine the octagon cut up into triangles with bases parallel to the base of the parallelogram, then the shearing motion preserves the lengths of these bases as well as the altitudes of the triangles, and thus their areas.[1] Hence, the area of the octagon in the rectangle is the same as that in the original parallelogram.

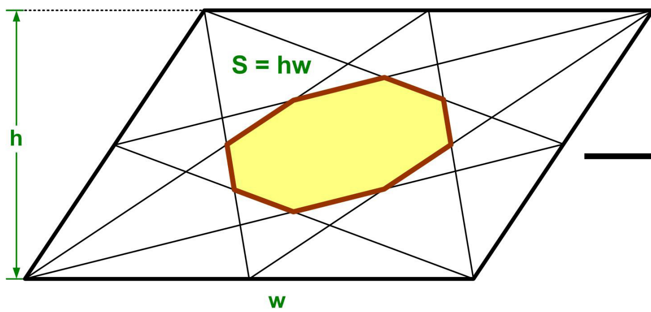


Figure 1

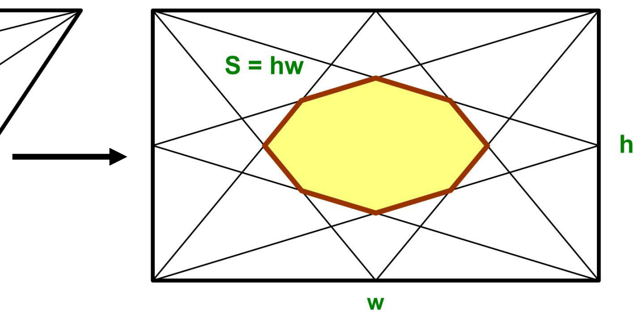


Figure 2

Next multiply the horizontal distances of the rectangle by a scale factor of *h/w*. This results in a square with sides of length *h* (Figure 3, Figure 4). If we think again of the triangle decomposition of the rectangle with bases parallel to the base of the rectangle, then the *h/w* scale factor reduces the lengths of the bases of the triangles, and thus their areas, by only that much, since the altitudes are unchanged. To restore the original area of any figure in the square, we multiply by the reciprocal *w/h*.

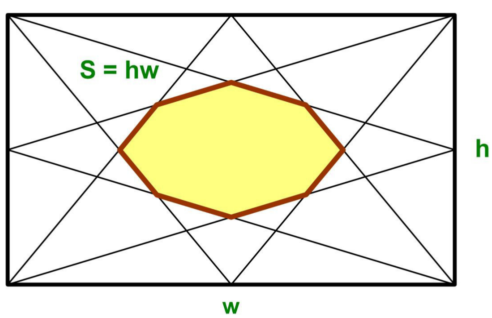


Figure 3

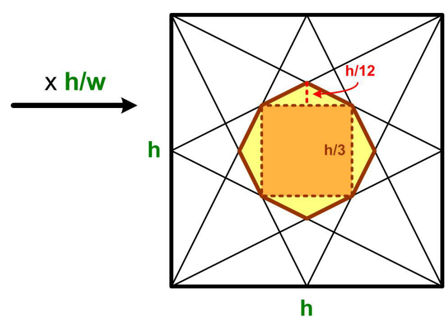


Figure 4

[1] A more detailed discussion of this idea is presented in my post "Quadrangle in Parallelogram" (http://josmfs.net/2020/06/27/quadrangle-in-parallelogram/).

From Figure 4 we can easily compute the area of the octagon as follows. All the four, similar (blue) right triangles in the square have the ratio of their long to short legs as 2 : 1. From this we conclude that the diagonal of the square inscribed in the octagon is 1/3 the diagonal of the large square and thus the side of the square has length $h/3$ (Figure 5). So its area is $h^2/9$. Half of the base of a yellow triangle is $h/6$, so the altitude must be half of that or $h/12$. Thus the area of each yellow triangle is $h^2/(6\cdot 12)$. Hence the area of the octagon is the area of the square plus the area of 4 yellow triangles or

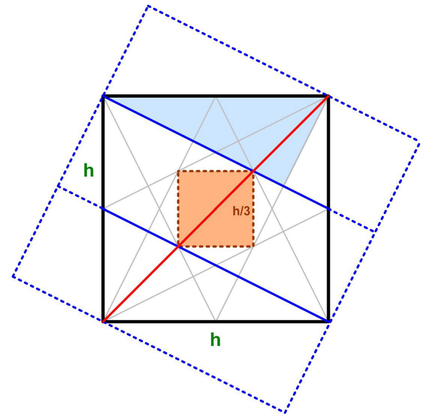


Figure 5

$$\text{Area of octagon} = h^2/9 + 4\,h^2/(6\cdot 12) = h^2\,(1/9 + 1/18) = h^2/6$$

Therefore the area of the original octagon, after multiplying by w/h, is

$$\text{Area of original octagon} = (w/h)\,h^2/6 = S/6$$

or one sixth the area of the parallelogram.

Olympiad Solution

I include the Olympiad solution as a direct image.

95. Denote the mid-points of the sides of the parallelogram $ABCD$ by K, L, M, N, as shown in Fig. 85, and the centre of the parallelogram by O. The segment joining vertex A with the mid-point L of side BC intersects the segment joining the mid-points K and M of AB and CD at point S and the diagonal BD at a point T. Then

(a) $OS = \frac{1}{2} OK$ since S is the centre of the parallelogram $ABLN$;

(b) $OT = \frac{1}{3} OB$, which can be proved as follows. Let us draw a segment joining the mid-point K of segment AB with the mid-point P of segment BL. Then $KP \parallel SL$; thus if Q is the point of intersection of segments KP and OB, then in triangle KOQ we have $OT = TQ$ and in triangle BTL we have $TQ = QB$.

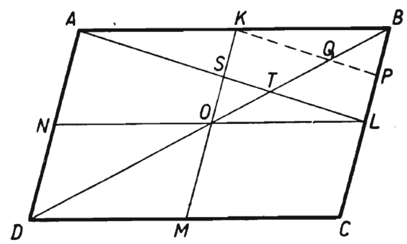


FIG. 85

This implies that the area of triangle SOT is equal to $\frac{1}{6}$ the area of triangle KOB.

An analogous reasoning applies to any other segment joining a vertex of the parallelogram with the mid-point of one of the opposite sides. Parts of those segments, such as ST, are the bases of eight triangles forming together an octagon (Fig. 86). The area of each of those triangles equals $\frac{1}{6}$ of the corresponding part of the parallelogram, whence the area of the octagon is equal to $\frac{1}{6}$ the area of the parallelogram.

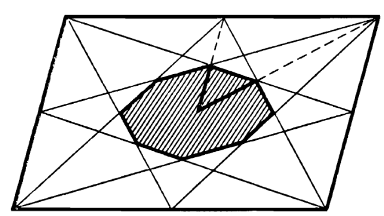


FIG. 86

References

[1] Straszewicz, S., *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads*, J. Smolska, tr., Popular Lectures in Mathematics, Vol.12, Pergamon Press, London, 1965 (Polish edition 1960). Problem p.116, solution p.189.