Circle Chord Problem

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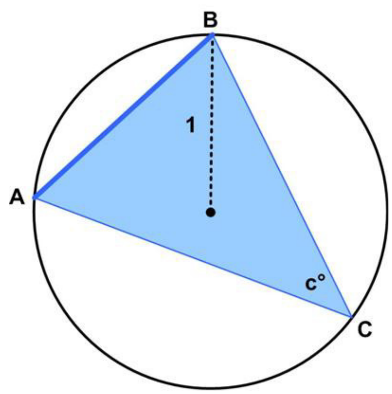

This is another nice puzzle from the Scottish Mathematical Council (SMC) Senior Mathematical Challenge of 2008 ([1]).

The triangle *ABC* is inscribed in a circle of radius 1. Show that the length of the side *AB* is given by 2 sin *c*°, where *c*° is the size of the interior angle of the triangle at *C*.

The diagram shows the case where C is on the same side of the chord AB as the center of the circle. There is a second case to consider where C is on the other side of the chord from the center.

My Solution

Case 1. If the vertex C is moved anywhere around the large arc of the circle from A to B, the value of c° remains the same, since it is an inscribed angle subtending the same (short) arc of the circle from A to B. Therefore the problem does not change if we move C to C' such that AC' is a diameter of the circle (Figure 1). But that means ABC' is a right triangle with hypotenuse 2, so that we immediately have

$$AB = 2 \sin c°.$$

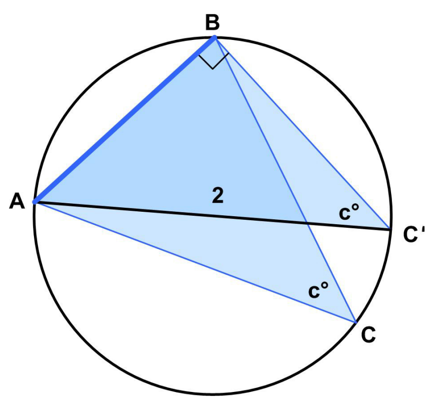

Figure 1

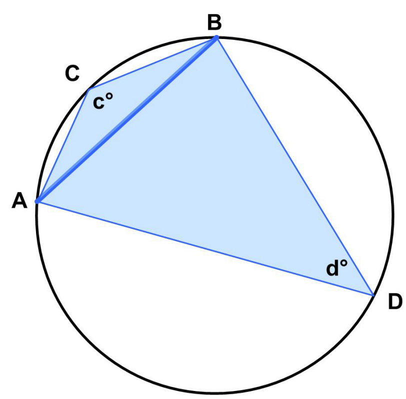

Figure 2

Case 2. If the vertex C is on the other side of the chord AB from the center, then

$$d° = (360° – 2c°)/2 = 180° – c°$$

is an inscribed angle that subtends AB as in Case 1 (Figure 2). Therefore,

$$AB = 2 \sin d° = 2 \sin (180° – c°) = 2(\sin 180° \cos c° – \sin c° \cos 180°) = 2(0 – \sin c° (–1)) = 2 \sin c°$$

SMC Solution

This is the SMC solution ([2]).

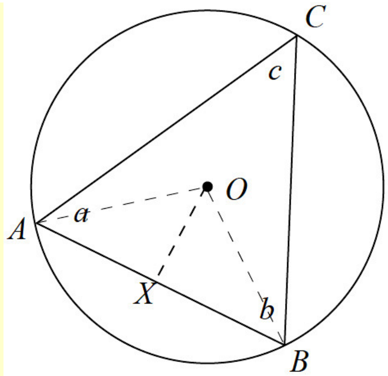

Figure 3

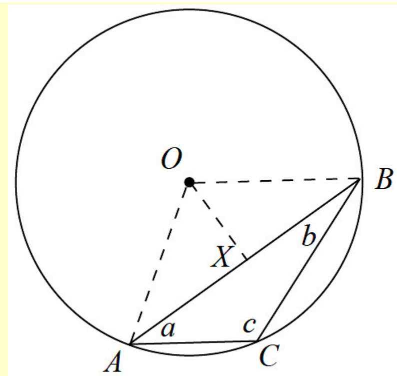

Figure 4

1. Let the centre of the circle be *O*, and let the interior angles at the vertices *A*, *B* and *C* be *a*, *b* and *c*, respectively. (Clearly, from the sine rule,[1] $AB = k \sin c$. It's a matter of determining the value of *k*).
2. Consider two situations: $\angle C$ is acute (Figure 3), and $\angle C$ is obtuse (Figure 4).
3. For both Figs: draw *OA*, *OB* (each length 1), and draw the perpendicular *OX*. Note that, since ΔAOB is isosceles, $AX = XB$ and $\angle AOX = \angle BOX$.
4. Figure 3: $\angle AOB = 2c$ (angle at centre is twice that at circumference from common chord – proof given below). Similarly, in Figure 4: $\angle AOB = 2\,(180 - c)$.
5. Figure 3: from triangle *AOB*, $AB = 2 \sin (½\angle AOB) = 2 \sin c$. Similarly, for Figure 4: from triangle *AOB*, $AB = 2 \sin (½\angle AOB) = 2 \sin (180 - c) = 2 \sin c$.

Conclusion The length of *AB* is $2 \sin c$ as required.

Proof of 4: Consider a chord *PQ* of a circle centre *O*, and any diameter *RS* which cuts the chord inside the circle, where *R* lies on the shorter arc between *P* and *Q*. Angles *SPR* and *SQR* are both right angles. Let $x = \angle PRS$, $y = \angle QRS$, $u = \angle POS$ and $v = \angle QOS$. Triangles *POR* and *QOR* are isosceles, so $u = 2x$ and $v = 2y$. The angle at the centre is $\angle POQ = u + v$, and the angle at *R*, on the circumference, is $\angle PRQ = x + y$. Thus, $\angle POQ = u + v = 2x + 2y = 2\,(x + y) = 2\angle PRQ$.

References

[1] "Senior Division: Problems 2" *Mathematical Challenge 2007−2008*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-2007-08-Q2.pdf)

[2] "Senior Division: Problems 2 Solutions" *Mathematical Challenge 2007−2008*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-2007-08-S2.pdf)

[1] JOS: I am not sure which "sine rule" they are referring to.