# Another Christmas Tree Puzzle

21 January 2022

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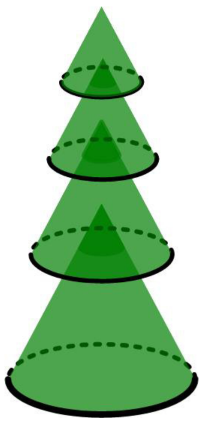

This is a belated Christmas puzzle from December 2019 MathsMonday.[1]

A Christmas tree is made by stacking successively smaller cones. The largest cone has a base of radius 1 unit and a height of 2 units. Each smaller cone has a radius 3/4 of the previous cone and a height 3/4 of the previous cone. Its base overlaps the previous cone, sitting at a height 3/4 of the way up the previous cone.

What are the dimensions of the smallest cone, by volume, that will contain the whole tree for any number of cones?

Recall that the volume of a cone is $\pi r^2 h/3$.

## My Solution

We begin by imagining a cone enveloping the Christmas tree as shown in Figure 1. The diagram suggests the tree cones may shrink in a linear fashion, but that is not clear a priori. At least we can hope they shrink at most linearly, and perhaps faster.

Figure 2 shows a cross section of the tree with the parameters of the problem used to establish the coordinates of the outer reach of the cones. It turns out that edges of the cones do shrink linearly, given by the equation $y = -{}^1/_6\, x + 1$, as we will show. This means when $y = 0$, $x = 6$, which is the height of the circumscribing cone. So the volume is $\pi r^2 h/3 = \pi\, 1^2\, 6/3 = 2\pi$.

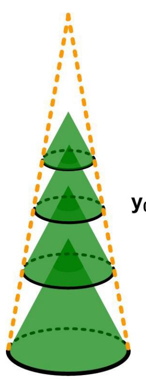

Figure 1

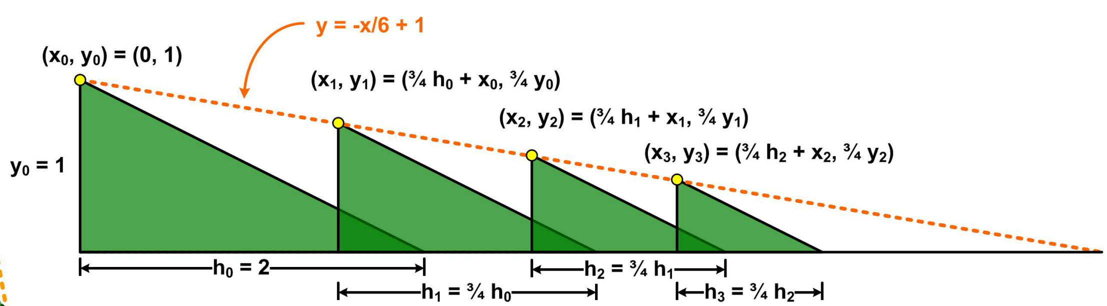



**Figure 2 Problem Parameterization**

The points in the figure follow from the terms of the problem, so we have the following sequence for the $x$ coordinates:

$$x_1 = x_0 + \frac{3}{4}h_0$$

$$x_2 = x_1 + \frac{3}{4}h_1 = x_0 + \frac{3}{4}h_0 + \left(\frac{3}{4}\right)^2 h_0$$

$$x_3 = x_2 + \frac{3}{4}h_2 = x_0 + \frac{3}{4}h_0 + \left(\frac{3}{4}\right)^2 h_0 + \left(\frac{3}{4}\right)^3 h_0$$

[1] https://twitter.com/MEIMaths/status/1206514304544985093 December 16, 2019

…

$$x_n = h_0\left(\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots+\left(\frac{3}{4}\right)^n\right)$$

since $x_0 = 0$. Now

$$\left(\frac{3}{4}\right)x_n - x_n = h_0\left(\left(\frac{3}{4}\right)^{n+1}-\frac{3}{4}\right)$$

So

$$x_n = 4h_0\left(\frac{3}{4}-\left(\frac{3}{4}\right)^{n+1}\right)=3h_0\left(1-\left(\frac{3}{4}\right)^n\right) \to 3h_0 = 6,\text{ as } n\to\infty, \qquad (*)$$

since $h_0 = 2$. And so the height of the enveloping cone is 6, as we claimed.

But we also claimed the cone edges descended linearly. First consider the $y$ coordinates from Figure 2:

$$y_1 = \frac{3}{4}y_0$$

$$y_2 = \frac{3}{4}y_1 = \left(\frac{3}{4}\right)^2 y_0$$

…

$$y_n = \frac{3}{4}y_{n-1} = \left(\frac{3}{4}\right)^n y_0 = \left(\frac{3}{4}\right)^n$$

since $y_0 = 1$.

Now compute $y = -{}^1/_6\, x_n + 1$, using the result from equation (*) and the fact that $h_0 = 2$,

$$y = -\frac{1}{6}x_n + 1 = -\frac{1}{6}\left(3h_0\left(1-\left(\frac{3}{4}\right)^n\right)\right)+1 = -1+\left(\frac{3}{4}\right)^n+1 = \left(\frac{3}{4}\right)^n = y_n$$

So the points $(x_n, y_n)$ do all line on the line $y = -{}^1/_6\, x + 1$.

# MathsMonday Solution

This MathsMonday problem came from an MEI Maths Item of the Month (December 2018), which also had the solution. Alas, I did not copy the solution when I first copied the problem (I did not want to know how it might be solved until I had a chance to solve it). In the meantime, MEI completely reformatted their website and broke all the previous links. Rather than list the problems explicitly, they buried them under a bunch of pedagogical categories under “Resources”, so that the problems had to be found via a search engine. However, the search engine could not find anything associated with the MathsMonday category for the problem, “A Level Mathematics: Pure Maths – Sequences and series.” In fact, the search engine seemed unable to find anything, no matter what I tried. Sigh.