Remainder Problem

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Here is a challenging problem from the 2021 Math Calendar ([1]).

Find the remainder from dividing the polynomial

$$x^{20} + x^{15} + x^{10} + x^5 + x + 1$$

by the polynomial

$$x^4 + x^3 + x^2 + x + 1$$

Recall that all the answers are integer days of the month.

Solution

Again we could just divide the polynomials to get the remainder, but we would rather apply some of the “short-cuts” from the “Polynomial Division Problem”.[1] Let

$$q(x) = x^{20} + x^{15} + x^{10} + x^5 + 1$$

and

$$p(x) = x^4 + x^3 + x^2 + x + 1.$$

Then $(x - 1)\, p(x) = x^5 - 1$, so the roots of $p(x) = 0$ are the 5^{th} roots of unity other than 1. They are given by

$$\alpha_k = e^{i2\pi k/5} \text{ for } k = 1, 2, 3, 4, \text{ where } \alpha_0 = 1.$$

From the division algorithm for polynomials, dividing $q(x)$ by $p(x)$ yields polynomials $m(x)$ and $r(x)$ such that

$$q(x) = m(x)\, p(x) + r(x) \quad \text{where} \quad \deg r(x) < \deg p(x) = 4$$

Just like in the “Polynomial Division Problem” we can write $q(x)$ as

$$q(x) = (x^5)^4 + (x^5)^3 + (x^5)^2 + x^5 + 1$$

and so for $k = 1, 2, 3, 4$,

$$q(\alpha_k) = (\alpha_k{}^5)^4 + (\alpha_k{}^5)^3 + (\alpha_k{}^5)^2 + \alpha_k{}^5 + 1 = 5$$

Therefore, for $k = 1, 2, 3, 4$,

$$5 = q(\alpha_k) = m(\alpha_k)\, p(\alpha_k) + r(\alpha_k) = m(\alpha_k)\cdot 0 + r(\alpha_k)$$

This means $s(x) = r(x) - 5$ is a polynomial of degree at most 3 but with 4 zeros, meaning $s(x) = 0$ has 4 roots. But it can have at most 3 roots, unless it is identically zero. That means the remainder $r(x)$ is the constant 5.

[1] https://josmfs.net/2021/08/14/polynomial-division-problem/

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2021: Your Daily epsilon of Math*, Rock Point, Quarto Publishing Group, New York, 2021. April