Existence Proofs II

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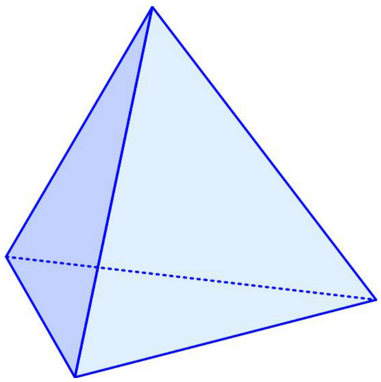

Futility Closet has another example ([1]) of an existence proof like their previous one ([2]) taken from Peter Winkler's 2021 *Mathematical Puzzles* (see my post "Existence Proofs" ([3])):

> Four bugs live on the four vertices of a regular tetrahedron. One day each bug decides to go for a little walk on the tetrahedron's surface. After the walk, two of the bugs have returned to their homes, but the other two find that they have switched vertices. Prove that there was some moment when all four bugs lay on the same plane.

My Solution

Label the bugs at the vertices A, B, C, and D as shown in Figure 1. Suppose bugs A and B are the ones that return to their homes, and that bugs C and D switch homes. Further, consider the triangle formed by A, B, and C and the plane that passes through these three points. At the beginning the configuration will look like that in Figure 1 and at the end it will look like that in Figure 3. So on D's way to the other vertex it must pass through the plane determined by A, B, and C (Figure 2).

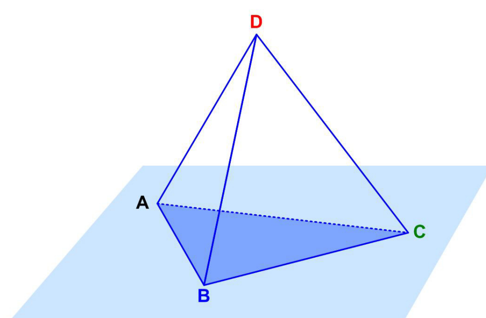

Figure 1

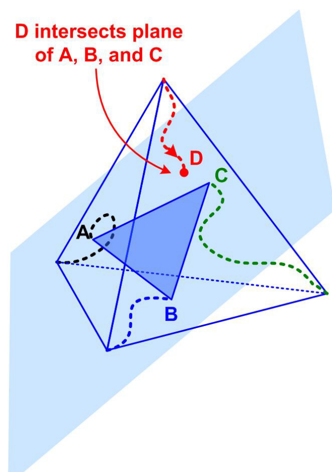

Figure 2

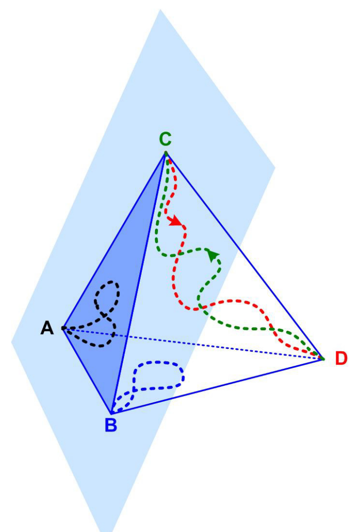

Figure 3

A more analytic proof of the situation might be to consider the unit normal to the moving plane determined by A, B, and C. A normal vector would be **AB x AC** where **AB** is the vector from A to B and **AC** is the vector from A to C. Let **n** be the unit vector determined by **AB x AC** (we assume the bugs never pairwise coincide so that these two vectors are never zero). Let d be the perpendicular distance between the plane and D. Then the vector $\mathbf{d} = d\,\mathbf{n}$ is the perpendicular vector from the plane to D. If d_0 is the initial distance between D and the plane in Figure 1, so that initially $\mathbf{d} = d_0\,\mathbf{n}$ (Figure 4), then being a regular tetrahedron means $\mathbf{d}$ is $-d_0\,\mathbf{n}$ (Figure 5) at the end in Figure 3. Since all the motions are continuous, as d ranges from d_0 to $-d_0$, there must be an instant when $d = 0$, which means D is in the plane of A, B, and C.

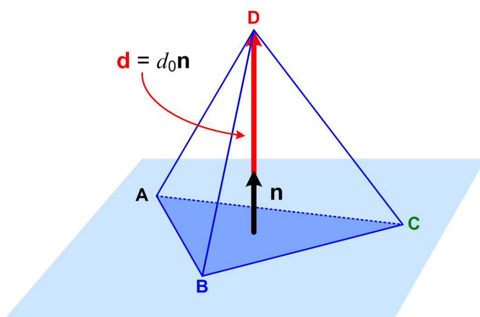

Figure 4

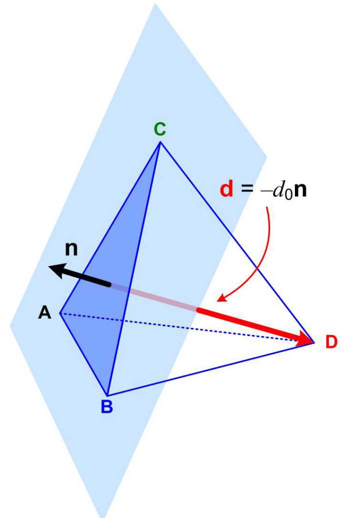

Figure 5

Futility Closet Solution

Futility Closet's solution is essentially mine, but with attention paid to the case I neglected where A, B, and C are collinear and do not determine a plane.

Call the bugs A, B, C, and D. In the initial position, when D looks at the other three bugs, she sees triangle ABC labeled clockwise, but in the final position she sees it labeled counterclockwise (or vice versa).[1] Since the bugs move continuously, some position must arise during the walk in which either D is on the plane ABC or ABC do not determine a plane. In the latter case they're collinear, which means they share a plane with any other point, including D.

The only way the three bugs A, B, and C can be collinear is if they all are on the same face of the tetrahedron. I guess I was assuming that would not be the case in general. Nevertheless, it should be considered in order to have a complete solution.

References

[1] "Exercise", *Futility Closet*, 26 October 2021 (https://www.futilitycloset.com/2021/10/26/exercise-3/)

[2] "State House", *Futility Closet*, 15 June 2021 (https://www.futilitycloset.com/2021/06/15/state-house/)

[3] Stevenson, James, "Existence Proofs", *Meditations on Mathematics*, 11 September 2021 (https://josmfs.net/2021/09/11/existence-proofs/)

[1] JOS: In my solution the orientation is captured in the order of the cross product **AB x AC** and consequently in the sign of the unit normal **n**. Thus the reverse orientation is given by the negative of the normal –**n**.