Polygon Rings

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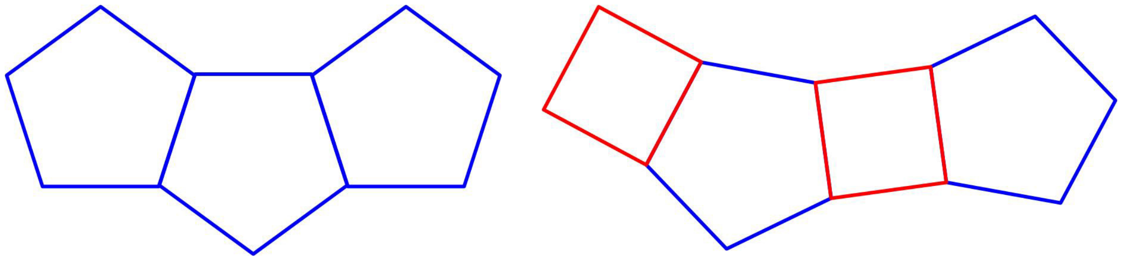

This is a nice geometric problem from the Scottish Mathematical Council (SMC) Senior Mathematical Challenge of 2008 ([1]).

Mahti has cut some regular pentagons out of card and is joining them together in a ring. How many pentagons will there be when the ring is complete?

She then decides to join the pentagons with squares which have the same edge length and wants to make a ring as before. Is it possible? If so, determine how many pentagons and squares make up the ring and if not, explain why.

My Solution

We can find the interior angles of the regular pentagon by moving a tangent vector around the figure back to its starting point. The vector will make 5 equal rotations at each vertex to make a total of 360°. Therefore the rotation at each vertex is 72°, and so the corresponding interior angle at the vertex is 180° – 72° = 108°.

Now consider the ring of regular pentagons. As shown in Figure 1, the inner edge of each pentagon will make an angle of 36° with the preceding pentagon's inner edge. If we can make a ring of n pentagons, then n 36° = 360°. So n = 10, which is an integer. Therefore 10 pentagons will make a ring.

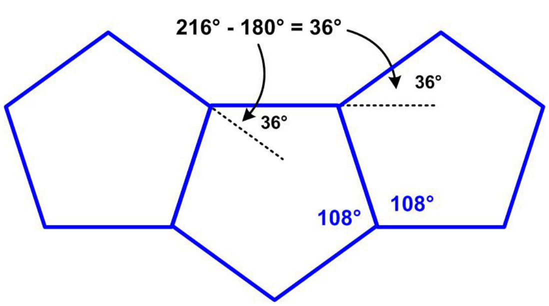

Figure 1

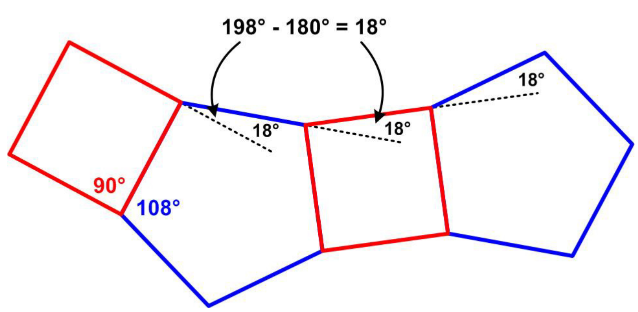

Figure 2

Now consider Figure 2. Each inner edge makes an angle of 18° with its predecessor. So if there is a ring of these regular polygons, then n 18° = 360° for some integer n. In this case n = 20. That means there are 10 square-pentagon pairs completing the ring and we don't end up with two adjacent pentagons or two adjacent squares in the ring.

SMC Solution

This is the SMC Solution ([2]).

If a_n° is the internal angle of a regular n-gon, then splitting it into identical isosceles triangles we see that $na_n + 360 = 180n$ so that $a_n = 180(1 - 2/n)$. So the internal angle of a regular pentagon is 108°. So the internal angle of the ring made out of pentagons is (360 − 2 × 108)° which is 144° which is the internal angle of a regular 10-gon. So there will be 10 pentagons in the ring.

For the pentagons and the squares, they will fit together in a ring if we can find a regular n-gon for some even number whose internal angle is b° where

$$b = 360 - (108 + 90) = 162 = 180\,(1 - 2/n).$$

So $n = 20$. Thus they do form a ring which contains 10 pentagons and 10 squares.

References

[1] "Senior Division: Problems 2" *Mathematical Challenge 2007−2008*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-2007-08-Q2.pdf)

[2] "Senior Division: Problems 2 Solutions" *Mathematical Challenge 2007−2008*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-2007-08-S2.pdf)