Two Curious Semicircles

4 February 2021

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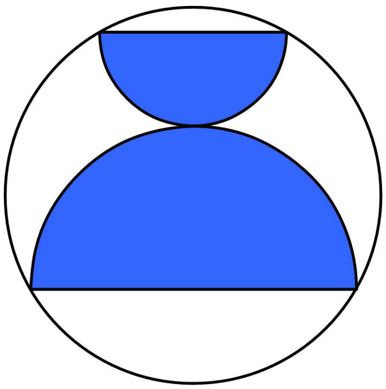

This is a nice brain tickling problem from Presh Talwalkar ([1]).

A circle contains two tangent semicircles whose diameters are parallel chords. If the circle has an area equal to 1, what is the combined area of the two semicircles?

My Solution

To get a quick answer apply the Polya principle when some parameters are not specified by choosing convenient values. In this case, assume the smaller radius r (Figure 1) shrinks to 0. Then the larger semicircle fills the upper half of the large circle and so coincides with half the area of the large circle. Therefore we know the answer must be one-half.

The problem implicitly wants us to *prove* the sizes of the radii don't matter. Using the notation of Figure 1, we want to show

$$\tfrac{1}{2}\pi\rho^2 = \tfrac{1}{2}\pi R^2 + \tfrac{1}{2}\pi r^2$$

or

$$\rho^2 = R^2 + r^2$$

Notice that the large red radii ρ are perpendicular, since they subtend the central angle of an inscribed 45° angle. We want a relationship that ties the three radii together and that is seen in the yellow triangle in Figure 1 using the Pythagorean Theorem:

$$2\rho^2 = (R + r)^2 + (R - r)^2 = 2R^2 + 2r^2$$

Done.

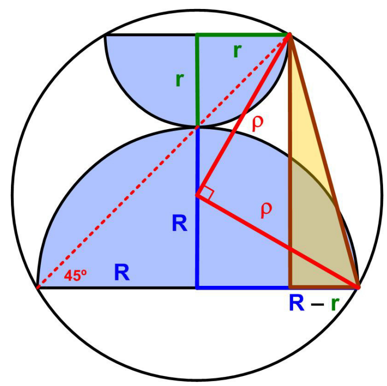

Figure 1

Talwalkar Solution

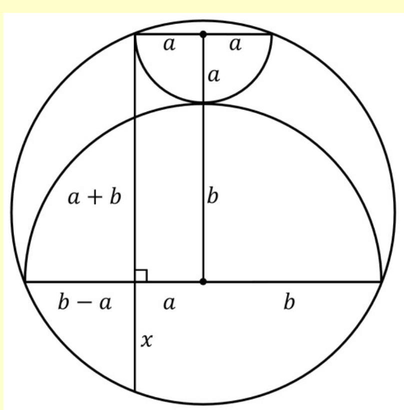

Figure 2

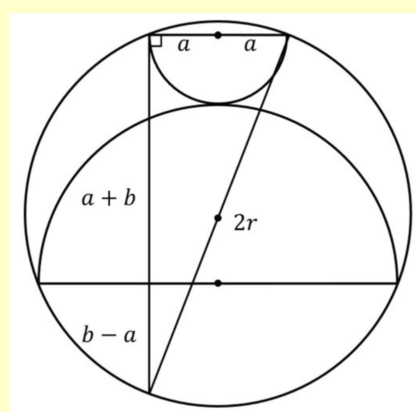

Figure 3

Suppose the top semicircle has a radius equal to *a* and the bottom semicircle a radius equal to *b*. Construct a line segment between the centers of the two semicircles and construct a parallel line segment from the left corner of the top semicircle to the radius of the bottom semicircle. This creates a rectangle with sides of a and $a + b$. Also extend the left line segment to be a chord of the circle, and let x be its distance below the diameter of the bottom semicircle [Figure 2].

We can solve for x by the intersecting chords theorem:[1]

$$x(a + b) = (b - a)(a + b)$$

$$x = b - a$$

We can then construct a right triangle with two legs of $2a$ and $a + b + x = 2b$. Since the right angle is an inscribed angle of the large circle, the chord opposite the right angle must be a diameter of the circle and have length equal to 2 times the radius of the large circle, call it $2r$ [Figure 3].

By the Gougu Theorem,[2] we have:

$$(2a)^2 + (2b)^2 = (2r)^2$$

$$4a^2 + 4b^2 = 4r^2$$

$$a^2 + b^2 = r^2$$

Now we can solve the problem. The area of the large circle is πr^2, and the area of the two semicircles is equal to:

$$\pi a^2/2 + \pi b^2/2 = \pi(a^2 + b^2)/2 = \pi r^2/2$$

Thus the two semicircles have an area equal to half of the circle! If the circle has an area equal to 1, then the two semicircles have an area equal to 1/2.

Special thanks this month to: Michael Anvari, Kyle, Mike Robertson. Thanks to all supporters on Patreon.[3]

References

Jobbings, A. (2011). 95.64 Two semicircles fill half a circle. The Mathematical Gazette, 95(534), 538-540. doi:10.1017/S0025557200003727 https://www.cambridge.org/core/journals/mathematical-gazette/article/9564-two-semicircles-fill-half-a-circle/6365F0D492DF2E682E71A891C782CAF6

The chalice problem https://www.youtube.com/watch?v=w87F9wy9BHM

Azimuth Project (John Carlos Baez) https://johncarlosbaez.wordpress.com/2014/01/12/geometry-puzzles/

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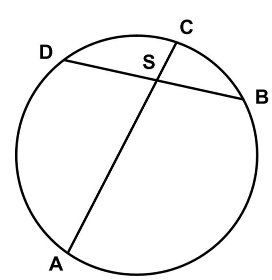
[1] JOS: For those, like me, who don't remember this less common theorem (though it is easy enough to prove) (https://en.wikipedia.org/wiki/Intersecting_chords_theorem):

> The *intersecting chords theorem* or just the chord theorem is a statement in elementary geometry that describes a relation of the four line segments created by two intersecting chords within a circle. It states that the products of the lengths of the line segments on each chord are equal. It is Proposition 35 of Book 3 of Euclid's *Elements*.
>
> More precisely, for two chords AC and BD intersecting in a point S the following equation holds:
>
> $$|AS| \cdot |SC| = |BS| \cdot |SD|$$



[2] JOS: Pythagorean Theorem. It is a Talwalkar idiosyncrasy to try to use the "original" authentic author of a theorem. This is in principle laudable, but in practice annoying, since it is basically somewhat arbitrary (parallel discoveries, author and/or date uncertain or unknown) and obscures standard nomenclature.

[3] http://www.patreon.com/mindyourdecisions

Folding proof by Greg Egan
http://math.ucr.edu/home/baez/mathematical/semicircle_puzzle_egan.gif

Cut The Knot
https://www.cut-the-knot.org/proofs/Semicircles.shtml

Comment. As I have mentioned in the past, I like to shy away from the less well-known theorems (at least to me), since most plane geometry results can be proved with a small set of elementary theorems, which most students have a chance of remembering some time after taking a plane geometry class.

This is also a Catriona Agg problem ([2]).

References

[1] Presh Talwalkar, "A Curious Fact About Two Semicircles", *Mind Your Decisions*, 2 February 2021. (https://mindyourdecisions.com/blog/2021/02/02/a-curious-fact-about-two-semicircles/)

[2] Catriona Agg, Aug 21, 2020 (https://twitter.com/Cshearer41/status/1296709944427896832)